Free probability and random matrices

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Abstract

In these lectures notes we will present and focus on free probability as a tool box to study the spectrum of polynomials in several (eventually) random matrices, and provide some applications.

Introduction

Free probability was introduced by Voiculescu as a non-commutative probability theory equipped with a notion of freeness that is very similar to independence in classical probability theory. Voiculescu [Voi91] showed at the early stage of free probability that freeness is describing the asymptotic behavior of a large class of random matrices, that is matrices whose eigenbasis are distributed very randomly, for instance with law invariant under unitary conjugation. Since then, free probability appeared as the natural set up to study the asymptotics of random matrices, at least when one is interested in polynomials of several (eventually random) matrices.

The basic object of interest in free probability, at least as far as random matrices are concerned, is related with the asymptotics of quantities

$$\tau_{X^N}(P) := \frac{1}{N}\mathrm{Tr}(P(X_1^N, \ldots, X_d^N))$$

where $(X_1^N, \dots, X_d^N)$ is a sequence of $N \times N$ (eventually) random matrices and P a polynomial in d non-commutative variables. Of course, such quantities not only depend on the eigenvalues of the matrices but also of the respective positions of the eigenvectors. Voiculescu considered the case where the latter is as random as possible, meaning for instance that the asymptotics of the random matrices under study are similar to those of $X_i^N = U_N^i D_i^N (U_i^N)^*$ with U_i^N independent unitary matrices following the Haar measure and $D_i^N = \mathrm{diag}(\lambda_j(X_i^N))$ is a diagonal matrix with entries given by the eigenvalues of X_i^N), and the spectral measure

$$L_{X_i^N} := \frac{1}{N} \sum_{j=1}^N \delta_{\lambda_j(X_i^N)}$$

converges in moments (that is $\int x^k dL_{X_i^N}(x)$ converges as N goes to infinity for each k. Then, Voiculescu showed that

$$\lim_{N \to \infty} \tau_{X^N}(P) = \tau_X(P)$$

converges almost surely as N goes to infinity and the limit can be described by the so-called notion of freeness. This notion is related with the notion of freeness on groups and can be used to effectively compute the limits. In particular, if P is a fixed self-adjoint polynomial, that is that $P(X_1^N, \dots, X_d^N) = P(X_1^N, \dots, X_d^N)^*$ for almost all realizations of $X_1^N, \dots, X_d^N$, then its spectral measure converges as

$$\begin{aligned} \int x^k dL_{P(X_1^N, \dots, X_d^N)}(x) &= \frac{1}{N} \mathrm{Tr}(P(X_1^N, \dots, X_d^N)^k) \\ &\to \tau_X(P^k) =: \int x^k dL_P(x)\,. \end{aligned}$$

In the cases under considerations the matrices are often asymptotically bounded so that $P(X_1^N, \dots, X_d^N)$ is a asymptotically bounded and hence the probability measure L_P has bounded support and is therefore uniquely determined by its moments. Therefore, free probability allows to analyze the convergence of the spectral measure of polynomials in several matrices. If the matrices possess the kind of randomness for the distribution of the eigenvectors that we described above, the limit will be described by freeness and free probability is the natural place where the analysis of this limit can be performed. For instance it allows to characterize the spectrum of the sum or the multiplication of random matrices, or to state general property such as the connectivity of the support of a large family of polynomials in random matrices. In fact, we shall see that it allows to study not only the spectral measure of polynomials in random matrices but more complicated objects. It allows as well to determine the asymptotics of extreme eigenvalues. The theory also developed to include the study of the fluctuations of the spectral measure [CD01, Gui02, MS06], large deviations [Voi00, BCG03] or Brownian motion. However we shall restrict ourselves to spectrum convergence in these notes.

There are many instances when one would like to study the spectrum of polynomial in several matrices, simply for instance when one is interested in a matrix which is given as such a polynomial.

The first example which comes to mind is the so-called Gaussian Wishart matrix $X = YY^*$ where $Y = (Y_1, \ldots, Y_N)$ are iid copies of a p dimensional Gaussian vector with covariance matrix Σ. Then if Z denote a $p \times N$ matrix with independent centered Gaussian entries with covariance one, then X has the same law as $Z\Sigma Z^*$ which is a polynomial in the matrix Z and the deterministic matrix Σ. Note that the law of Z is the same as the law of UZV where U, V are deterministic unitary (resp. orthogonall) matrices. Hence, such Wishart matrices naturally pertain to the domain of application of free probability when their size goes to infinity and indeed the spectrum of Y can be described by its tools. Eventhough the spectrum of Wishart matrices was studied without using such ideas, they appear to give a general and unified analysis [BG09].

Another natural question is for instance, being given two sequences of random matrices A and B, to understand the spectrum of its sum $A + B$. This of course also depends on their respective eigenbasis as if A and B have the same eigenvectors, the eigenvalues will simply be the sum of the eigenvalues of A and B, whereas if the eigenbasis of A and B are as much independent as possible in the sense that one is the "random rotation" of the other, then the spectrum will be described asymptotically by free convolution.

In some sense, in the case of non-normal matrices, it is even natural to study the spectrum of a matrix X by considering it together with its adjoint. If X is normal, that is can be decomposed as $X = UDU^*$ with D a diagonal matrix whose entries are the eigenvalues of X, and U a unitary matrix, X and X^* commute. To retrieve the spectrum $(\lambda_1, \ldots, \lambda_N)$ of X, and for instance its empirical distribution, it may be wise to consider

$$\mathrm{Tr}(X^p(X^*)^k) = \sum \lambda_i^p \bar{\lambda}_i^k$$

to characterize the spectral measure by moments. In the case of non-normal matrices, that is matrices which do not commute with their adjoint, it turns out that Green formula describes the spectral measure in terms of the spectral measure of $(z-X)(z-X)^*$ for $z \in \mathbb{C}$. Even though free probability theory does not provide all the tools to prove convergence of the spectrum of non-normal matrices it permits to guess the asymptotics in terms of the so-called Brown measure. Indeed, free probability set up is mainly designed to handle smooth functions of matrices such as polynomials whereas it is well known that the spectrum of non-normal matrices can be very unstable, leading to the notion of pseudo-spectrum ; changing only one entry by a very tiny number can change drastically the spectrum. However, if one forget this instability, or somehow neglect a singularity, free probability allows to predict and study the limit of the spectrum of non-normal matrices.

As we have emphasized earlier, free probability deals with matrices whose eigenbasis are very randomly chosen, at least one with respect to the other. This applies to many different models of random matrices, but not to all. For instance, it does not apply to covariance matrices of a Erdös-Rényi graph, that is a symmetric matrix with independent entries above the diagonal which vanish with probability $1 - p/N$ and equal one otherwise, or more generally with moments which do not vanish sufficiently fast. Another question that can not be solved by free probability is to analyze the spectrum of $A + PBP^*$ where P is a uniformly chosen permutation matrix. Then, it turns out that moments are not sufficient to describe the joint laws of such matrices but a notion of distribution of traffics [Mal11a] can be defined. Some of the apparatus of free probability can be generalized to this setting.

1 Random matrices and freeness

Free probability appears as the natural setting to study the asymptotics of traces of words in several (possibly random) matrices.

Adopting the point of view that traces of words in several matrices are fundamental objects is fruitful because it leads to the study of some general structure such as freeness ; freeness in turns simplifies the analysis of convergence of moments. The drawback is that one needs to consider more general objects than empirical measures of eigenvalues converging towards a probability measure, namely, traces of noncommutative polynomials in random matrices converging towards a linear functional on such polynomials, called a tracial state. Analysis of such objects is then achieved using free probability tools.

1.1 Non-commutative laws and freeness

During this course, we shall call non-commutative law of d self-adjoint non-commutative variables any linear form τ_X defined on the set $\mathbb{C}\langle X_1, \ldots, X_d\rangle$ of polynomial functions of d self-adjoint variables and with value in $\mathbb{C}$ which can be constructed thanks to a sequence $(X_1^N, \ldots, X_d^N)$ of $N \times N$ self-adjoint random matrices by

$$\tau_X(P) = \lim_{N\to\infty} \tau_{X^N}(P)$$

where

$$\tau_{X^N}(P) = \mathbb{E}[\frac{1}{N}\mathrm{Tr}(P(X_1^N, \ldots, X_d^N))] \,.$$

We shall call τ_{X^N} the empirical distribution of the matrices X^N. If $d = 1$, as $\tau_{X^N}(P) = \mathbb{E}[N^{-1}\sum_{i=1}^N P(\lambda_i)] = \mathbb{E}[\int P(x) dL_{X^N}(x)]$ if λ_i denote the eigenvalues of X^N, we conclude that τ_{X^N} is the mean spectral distribution. Hence, the space of non-commutative laws in one self-adjoint variable is simply the set of probability measures on $\mathbb{R}$.

We see that any $\tau = \tau_X$ which can be constructed as above are linear forms on $\mathbb{C}\langle X_1, \dots, X_d\rangle$ which satisfy

– A positivity property : we may endow $\mathbb{C}\langle X_1, \dots, X_d\rangle$ with the involution

$$\left(\sum z_p X_{i_1^p} \cdots X_{i_{n_p}^p}\right)^* = \sum \bar{z}_p X_{i_{n_p}^p} X_{i_{n_p-1}^p} \cdots X_{i_1^p} .$$

Then

$$\tau(PP^*) \geq 0 \qquad \forall P \in \mathbb{C}\langle X_1, \dots, X_d\rangle . \tag{1}$$

– A mass condition

$$\tau(1) = 1 . \tag{2}$$

– A tracial property

$$\tau(PQ) = \tau(QP) \qquad \forall P, Q \in \mathbb{C}\langle X_1, \dots, X_d\rangle$$

It is still an open question whether any linear form τ satisfying the above properties can be constructed as the limit of empirical distribution of matrices ; this is the general definition of the law of non-commutative variables.

The space of non-commutative laws thus appear as a kind of natural generalization to the space of probability measures. It will be throughout endowed with a notion of weak convergence : If τ_n is a sequence of laws of d self-adjoint non-commutative variables then τ_n converges towards τ off

$$\lim_{n\to\infty} \tau_n(P) = \tau(P) \qquad \forall P \in \quad \mathbb{C}\langle X_1, \dots, X_d\rangle .$$

We say that matrices $(X_1^N, \dots, X_d^N)$ converges in law towards τ if their empirical distribution τ_{X^N} converges weakly towards τ.

The notion of freeness gives an algebraic relation between moments. We say that random variables $X = (X_1, \dots, X_p)$ and $Y = (Y_1, \dots, Y_d)$ with joint law τ are free iff for any polynomials $P_1, \dots, P_k \in \mathbb{C}\langle X_1, \dots, X_p\rangle$

and $Q_1, \ldots, Q_k \in \mathbb{C}\langle X_1, \ldots, X_d \rangle$ so that $\tau(P_i(X))$ and $\tau(Q_i(Y))$ vanish, we have

$$\tau(P_1(X)Q_1(Y)P_2(X)\cdots P_k(X)Q_k(Y)) = 0\,. \tag{3}$$

It is easy to check by induction over the degree of polynomials that this relation defines uniquely joint moments if the marginals are known. Indeed, by definition for any polynomial P, Q

$$\tau(P(X)Q(Y)) = \tau(P(X))\tau(Q(Y))$$

is given by the marginals. More generally for any polynomial P_i, Q_i

$$\tau(P_1(X)Q_1(Y)P_2(X)\cdots P_k(X)Q_k(Y))$$
$$-\tau((P_1(X) - \tau(P_1(X))I)\cdots(Q_k(Y) - \tau(Q_k(Y))I))$$

is given by a sum of products of traces of polynomials which have (total) degree strictly smaller than $P_1(X)Q_1(Y)P_2(X)\cdots P_k(X)Q_k(Y)$. Hence, by induction we can compute all the moments since by freeness $\tau((P_1(X) - \tau(P_1(X))I)\cdots(Q_k(Y) - \tau(Q_k(Y))I))$ vanishes.

Two sequences of $N \times N$ matrices $(X_i^N, 1 \le i \le p)$ and $(Y_i^N, 1 \le i \le d)$ is called *asymptotically free* if they converges in law as N goes to infinity to free random variables $(X_i, 1 \le i \le p)$ and $(Y_i, 1 \le i \le d)$. In other words, for all $P \in \mathbb{C}\langle X_1, \ldots, X_p, Y_1, \ldots, Y_d \rangle$

$$\lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\mathrm{Tr}(P(X_1^N, \ldots, X_p^N, Y_1^N, \ldots, Y_d^N))] = \tau(P)$$

and $X = (X_1, \ldots, X_p)$ and $Y = (Y_1, \ldots, Y_d)$ with joint law τ are free.

2 Wigner matrices are asymptotically free

Let us consider Wigner matrices, that is self-adjoint matrices with independent entries above the diagonal. We assume that the law of the entries is independent of N and with all the moments finite. They can be either real or complex valued, and are centered.

Theorem 2.1 *Let X_i^N, $1 \le i \le d$, be a family of Wigner matrices. Assume that for all $k \in \mathbb{N}$,*

$$\sup_{N\in\mathbb{N}} \sup_{1\le i\le d} \sup_{1\le m\le \ell\le N} E[|X_i^N(m,\ell)|^k] \le c_k < \infty\,, \tag{4}$$

that $(X_i^N(m,\ell), 1 \le m \le \ell \le N, 1 \le i \le p)$ are independent, and that $E[X_i^N(m,\ell)] = 0$ and $E[|X_i^N(m,\ell)|^2] = 1$.

Then, the empirical distribution $\hat\tau_N := \tau_{\{\frac{1}{\sqrt{N}}X_i^N\}_{1\le i\le d}}$ of $\{\frac{1}{\sqrt{N}}X_i^N\}_{1\le i\le d}$ converges almost surely and in expectation to the law of d free semi-circular variables that is free variables with marginal distribution given by the semi-circle law

$$\sigma(dx) = \sqrt{4-x^2}dx/\pi\,.$$

Proof.

We shall prove that if we represent a monomial $q(X) = X_{i_1}\cdots X_{i_k}$ by k colored ordered points (called the points of type q) so that the first has color i_1, the second i_2 etc till the last which has color i_k, then

$$\begin{aligned}\tau(q) &= \lim_{N\to\infty} \mathbb{E}[\frac{1}{N}\mathrm{Tr}(q(\frac{X_1^N}{\sqrt{N}},\ldots,\frac{X_d^N}{\sqrt{N}}))] \\ &= \#\{\text{ non-crossing pair partitions of the points of type } q \\ &\qquad\quad \text{with one color blocks}\}\end{aligned}$$

The right hand side is the number of non-crossing pair partitions with mono-colored blocks, that is the number of partitions of $1,\ldots,k$ so that if $B_1 = (p_1,p_2)$ and $B_2 = (q_1,q_2)$ are two blocks of this partition, $p_1 < p_2$ and $q_1 < q_2$, they are mono-colored (that is $i_{p_1} = i_{p_2}$ as well as $i_{q_1} = i_{q_2}$) and non-crossing

$$p_2 < q_1 \qquad \text{or} \qquad p_1 < q_1 < q_2 < p_2\,.$$

This is the law of free semicircular variables. Indeed, first note that the moments of only one variable are given by the numbers of non-crossing pair partitions, which are given by the Catalan numbers. We leave the reader to check that they are also equal to the moments of the semi-circle law. Secondly, it is enough by linearity to prove the freeness conditions for monomials. But, if we decompose q as $q(X) = q_1(X_{i_1})q_2(X_{i_2})\cdots q_r(X_{i_r})$ with $i_j \neq i_{j+1}$ then

$$\tau((q_1(X_{i_1}) - \sigma(q_1))(q_2(X_{i_2}) - \sigma(q_2))\cdots(q_r(X_{i_r}) - \sigma(q_r)))$$

is the number of non crossing pair partitions so that the sets of points $S_{i+1} = \{\sum_{j=1}^{i} \mathrm{deg} q_j + 1, \ldots, \sum_{j=1}^{i+1} \mathrm{deg} q_j\}, 1 \leq i \leq r$ have pairings with each other. But we claim that there is no such non-crossing pair partitions. Indeed, let us consider the pairing of the first point. Either the other point of the pairing is inside the first set of points S_1 and then we replace the first point by its right neighbor. We can continue till either S_1 has no matching with the other points, in which case we are done, or there is one. In which case we can focus on the points $\{s, \ldots, t\}$ in the interior of this block. In the interior of $\{s, \ldots, t\}$ there are at least points of two different colors. We can continue until there is exactly two different colors in the points considered in which case the conclusion follows.

To prove the convergence, we expand the trace of $q = X_{i_1}\cdots X_{i_k}$as

$$\mathbb{E}[\frac{1}{N^{\frac{k}{2}+1}}\mathrm{Tr}(q(\frac{X_1^N}{\sqrt{N}}, \ldots, \frac{X_d^N}{\sqrt{N}}))] = \frac{1}{N^{\frac{k}{2}+1}} \sum_{j_1,\ldots,j_k=1}^{N} \mathbb{E}[X_{i_1}(j_1 j_2)\cdots X_{i_k}(j_k, j_1)]$$

To any set of indices $\{j_1, \ldots, j_k\}$ we associate the connected graph $G = (V, E)$ so that $E = \{j_q, j_{q+1}\}, 1 \leq q \leq k$ with the convention $j_{k+1} = j_1$. The term $\mathbb{E}[X_{j_1 j_2}\cdots X_{j_k j_1}]$ vanishes unless the edges of G appear at least with multiplicity 2 as non-oriented edges by independence and centering. But since G is connected, if $|E|$ is the number of different non-oriented edges in G and $|V|$ the numbers of distinct vertices, we have

$$|V| \leq |E| + 1 \leq \frac{k}{2} + 1\,.$$

Hence, there are at most $\frac{k}{2} + 1$ different vertices, that is at most $N^{\frac{k}{2}+1}$ choices of the indices. As we divide the sum by $N^{\frac{k}{2}+1}$, we see that graphs with $|V| \leq \frac{k}{2}$ will not contribute asymptotically. Hence, only graphs G so that $|V| = |E| + 1$ contribute, that is trees. Moreover, each edge has multiplicity two. Furthermore, there is a connected path $i_p \to i_{p+1}$ around this tree. Unfolding this path gives a bijection between the tree and a non-crossing pair partition. Finally, the graph contributes only if the edges which are paired correspond to the same matrix, that is if the non-crossing pair partition pairs only points of the same color.

Theorem 2.1 generalizes to the case of polynomials including some deterministic matrices.

Theorem 2.2 *Let $\mathbf{D}^N = \{D_i^N\}_{1\le i\le d}$ be a sequence of Hermitian deterministic matrices and let $\mathbf{X}^N = \{X_i^N\}_{1\le i\le d}$, $X_i^N : \Omega \to \mathcal{H}_N^{(\beta)}$, $1 \le i \le d$, be Wigner matrices satisfying the hypotheses of Theorem 2.1. Assume that*

$$D := \sup_{k\in\mathbb{N}} \max_{1\le i\le d} \sup_N \frac{1}{N} \mathrm{Tr}((D_i^N)^k)^{\frac{1}{k}} < \infty\,, \tag{5}$$

and that the law $\tau_{\mathbf{D}^N}$ of $\mathbf{D}^N$ converges to a noncommutative law μ. Then,the noncommutative variables $\frac{1}{\sqrt{N}}\mathbf{X}^N$ and $\mathbf{D}^N$ are asymptotically free. In particular, the empirical distribution $\tau_{\frac{1}{\sqrt{N}}\mathbf{X}^N,\mathbf{D}^N}$ converges almost surely and in expectation to the law of $\{\mathbf{X},\mathbf{D}\}$, $\mathbf{X}$ and $\mathbf{D}$ being free, $\mathbf{D}$ with law μ and $\mathbf{X}$ being d free semi-circular variables.

Theorem 2.2 can be proved as Theorem 2.1 if the D_i^N are diagonal matrices. However, this strategy is not straightforward to follow otherwise. The proof is give in [AGZ10, Section 5] by showing an approximate integration by parts formula. In some sense, Theorem 2.2 is a universality result and shows that Wigner matrices whose entries have enough finite moments have the same asymptotics as Gaussian. Gaussian matrices have the particularity that their law is invariant by multiplication $X^N \to U X^N U^*$. The next result shows that this is the real essence of freeness.

We now consider conjugation by unitary matrices following the Haar measure dU on the set of $N \times N$ unitary matrices.

Theorem 2.3 *Let $\mathbf{D}^N = \{D_i^N\}_{1\le i\le p}$ be a sequence of Hermitian (eventually random) $N \times N$ matrices. Assume that their empirical distribution converges to a noncommutative law μ. Assume also that there exists* *a deterministic $D < \infty$ such that for all $k \in \mathbb{N}$, all $N \in \mathbb{N}$,*

$$\frac{1}{N}\mathrm{Tr}((D_i^N)^{2k}) \leq D^{2k},\ a.s.$$

Let $\mathbf{U}^N = \{U_i^N\}_{1\leq i\leq p}$ be independent unitary matrices with Haar law $\rho_{U(N)}$, independent from $\{D_i^N\}_{1\leq i\leq p}$. Then, the matrices $\{U_i^N, (U_i^N)^*\}_{1\leq i\leq p}$, and the matrices $\{D_i^N\}_{1\leq i\leq p}$, are asymptotically free. For all $i \in \{1,\dots,p\}$, the limit law of $\{U_i^N, (U_i^N)^*\}$ is given by

$$\tau((UU^*-1)^2) = 0, \quad \tau(U^n) = \tau((U^*)^n) = \mathbf{1}_{n=0}\,.$$

This theorem can easily be deduced from the previous one by noticing that unitary matrices following the Haar measure are approximately smooth functions of GUE matrices. Indeed, both U following the Haar measure and X following the GUE law are normal matrices and thus can be decomposed as $U = VD_UV^*$ and $X = \tilde{V}D_X\tilde{V}^*$ where $V, \tilde{V}$ follow the Haar measure on the unitary group. D_U and D_X are diagonal matrices whose entries are the eigenvalues $(e^{i\theta_j})_{1\leq i\leq j}$ of U and $(\lambda_i)_{1\leq i\leq N}$ of X so that $j\to\theta_j$ and $j\to\lambda_j$ are increasing. We can couple these two matrices so that $V = \tilde{V}$ and moreover consider the map f_N which send the ordered eigenvalues of X onto the eigenvalues $e^{i\theta_j}$ of U chosen with increasing θ_j's. As shown in [CM12], if we put

$$F_X(t) = N^{-1}\sum_{j=1}^N 1_{\lambda_j\leq t} \qquad F_U(t) = N^{-1}\sum_{j=1}^N 1_{\theta_j\leq t}$$

then we can take $f_N(t) = \exp(iF_U^{-1}\circ F_X(t))$ which converges almost surely towards a deterministic smooth function. Hence, we can always approximate polynomials in the variables U_i^N by polynomials in G_i^N and therefore deduce the last theorem from Theorem 2.1.

We have the following corollary, which shows that freeness can be seen as a consequence of independence of the respective eigenbasis.

Corollary 2.4 *Let $\{D_i^N\}_{1\leq i\leq p}$ and $\{\tilde{D}_j^N\}_{1\leq j\leq d}$ be sequences of uniformly bounded self-adjoint matrices which converge in law towards τ_D*

and $\tau_{\tilde{D}}$ *respectively. Let* U^N *be an independent unitary matrix following the Haar measure. Then, the noncommutative variables* $\{U^N D_i^N (U^N)^*\}_{1\le i\le d}$ *and* $\{\tilde{D}_j^N\}_{1\le j\le p}$ *are asymptotically free, the law of the marginals being given by* τ_D *and* $\tau_{\tilde{D}}$.

Assuming the convergence of the joint law of $\{D^N, \tilde{D}^N\}$, the Corollary is just due to $\tau(U) = \tau(U^*) = 0$ since

$$P(U^N D_i^N (U^N)^*, 1 \le i \le p) = U^N P(D_i^N, 1 \le i \le p)(U^N)^*$$

so that monomials in $\{U^N D_i^N (U^N)^*\}_{1\le i\le p}$ and $\{\tilde{D}_j^N\}_{1\le j\le p}$ can be written as products of monomials in the D's interlaced by one unitary U^N or its adjoint $(U^N)^*$, the asymptotic freeness of $\{U^N D_i^N (U^N)^*\}_{1\le i\le p}$ and $\{\tilde{D}_j^N\}_{1\le j\le p}$ is a consequence of the asymptotic freeness of $\{U_i^N, (U_i^N)^*\}_{1\le i\le p}$, and the matrices $\{D_i^N\}_{1\le i\le p}, \{\tilde{D}_j^N\}_{1\le j\le d}$. A closer look shows that we do not need to have convergence of the joint law but only of the marginals.

2.1 Strong asymptotic freeness

Until this point, free probability was introduced as a tool to study the asymptotics of polynomials in several matrices, which gives information on "macroscopic quantities" such as the empirical measure of the eigenvalues. However, it is well known that for many models of random matrices, the extreme eigenvalues stick to the bulk, that is converge to the boundary of the support of the limiting spectral measure. It turns out that this phenomenon can be extended to all polynomials in random matrices. Strong asymptotic freeness precisely means that not only the empirical distribution of the matrices converge to the law of free variables, but that also the operator norm of these matrices converge to their free limit. Namely, if $\mathbf{D}^N = \{D_i^N\}_{1\le i\le p}$ be a sequence of Hermitian (eventually random) $N \times N$ matrices. Assume that their empirical distribution converges to a noncommutative law τ. Assume also that there exists a deterministic $D < \infty$ such that for all $k \in \mathbb{N}$, all $N \in \mathbb{N}$,

$$\frac{1}{N}\mathrm{Tr}((D_i^N)^{2k}) \le D^{2k}, \ a.s.$$

Let $\mathbf{U}^N = \{U_i^N\}_{1\le i\le p}$ be independent unitary matrices with Haar law ρ, independent from $\{D_i^N\}_{1\le i\le p}$. Let $\mathbf{X}^N = \{X_i^N, 1 \le i \le d\}$ be independent GUE matrices. Let P be a polynomial in $(\mathbf{D}^N, \mathbf{U}^N, \mathbf{X}^N)$ and denote by $\|P(\mathbf{D}^N, \mathbf{U}^N, \mathbf{X}^N)\|$ the operator norm of $P(\mathbf{D}^N, \mathbf{U}^N, \mathbf{X}^N)$ given by

$$\|P(\mathbf{D}^N, \mathbf{U}^N, \mathbf{X}^N)\| = \lim_{p\to\infty} \left(\mathrm{Tr}(P(\mathbf{D}^N, \mathbf{U}^N, \mathbf{X}^N)P(\mathbf{D}^N, \mathbf{U}^N, \mathbf{X}^N)^*)^p\right)^{1/2p}$$

On the other hand, let $(\mathbf{D}, \mathbf{U}, \mathbf{X})$ be free variables with marginal distribution τ, free unitaries and free semi-circle. We let

$$\|P(\mathbf{D}, \mathbf{U}, \mathbf{X})\| = \lim_{p\to\infty} \tau((P(\mathbf{D}, \mathbf{U}, \mathbf{X})P(\mathbf{D}, \mathbf{U}, \mathbf{X})^*)^p)^{1/2p}$$

Then we have the following result, proved for Gaussian matrices in the breakthrough paper [HT05], for Wigner matrices in [CDM07, And11], for joint deterministic and GUE [Mal11b] and under this form in [CM12] :

Theorem 2.5 *Assume that for each polynomial P*

$$\lim_{N\to\infty} \|P(\mathbf{D}^N)\| = \|P(\mathbf{D})\|$$

Then, for any polynomial P

$$\lim_{N\to\infty} \|P(\mathbf{D}^N, \mathbf{U}^N, \mathbf{X}^N)\| = \|P(\mathbf{D}, \mathbf{U}, \mathbf{X})\| \qquad a.s.$$

The proof of such a result requires finer estimate than just finite moments : the idea developed in [HT05] (and then in [CDM07, And11]) is to realize that the spectrum of a polynomial P is related with that of a larger matrix which is linear in the random matrices (the so-called linear trick) and then study the Stieljes transform of this matrix by using the so-called loop equations. The extension to unitary matrices is performed in [CM12] by approximating unitary matrices by polynomials in GUE matrices.

Interestingly enough, free probability also can be used to study eigenvalues which do not stick to the bulk. Indeed, in [CDMFF11], the extreme eigenvalues of $M = A + X$ were studied with X a Wigner matrix with entries satisfying Poincaré inequality and A a deterministic matrix with converging spectral distribution but also with some outliers, that is eigenvalues which do not stick to the bulk. It was proved that the positions of the outliers of M, when it has some, can be described by the subordination function which describes free convolution, see section **??**.

2.2 Convergence of the spectral measure of polynomials in random matrices

We next specialize the previous theorem to study the spectrum of a polynomial in uniformly bounded deterministic matrices $\mathbf{D}^N$, and random matrices $\mathbf{U}^N$ following the Haar measure and $\mathbf{X}^N$ being independent Wigner matrices. By the previous point their empirical distribution converges towards the law of free variables $\mathbf{D}, \mathbf{U}, \mathbf{X}$ which are bounded. Hence, if P is a self-adjoint polynomial function, $P(\mathbf{D}, \mathbf{U}, \mathbf{X})$ is bounded uniformly and in particular the law μ_P of $P(\mathbf{D}, \mathbf{U}, \mathbf{X})$ is determined by its moments :

$$\int x^k d\mu_P(x) := \tau((P(\mathbf{D}, \mathbf{U}, \mathbf{X}))^k)\,.$$

From the previous section we infer the following result :

Corollary 2.6 *Let $\mathbf{D}^N = \{D_i^N\}_{1\le i\le p}$ be a sequence of Hermitian (eventually random) $N \times N$ matrices. Assume that their empirical distribution converges to a noncommutative law τ. Assume also that there exists a deterministic $D < \infty$ such that for all $k \in \mathbb{N}$, all $N \in \mathbb{N}$,*

$$\frac{1}{N}\mathrm{Tr}((D_i^N)^{2k}) \le D^{2k},\ a.s.$$

Let $\mathbf{U}^N = \{U_i^N\}_{1\le i\le p}$ be independent unitary matrices with Haar law ρ, independent from $\{D_i^N\}_{1\le i\le p}$. Let $\mathbf{X}^N = \{X_i^N, 1 \le i \le d\}$ be independent Wigner matrices with entries with finite moments. Let P be a self-adjoint polynomial in $(\mathbf{D}^N, \mathbf{U}^N, \mathbf{X}^N)$. Then the spectral measure *of $P(\mathbf{D}^N, \mathbf{U}^N, \mathbf{X}^N)$ converges towards the law μ_P of $P(\mathbf{D}, \mathbf{U}, \mathbf{X})$ where $\mathbf{D}, \mathbf{U}, \mathbf{X}$ are free with marginal distribution given respectively by τ, free unitaries and free semi-circle distribution, whereas the extreme eigenvalues of $P(\mathbf{D}^N, \mathbf{U}^N, \mathbf{X}^N)$ converges towards the extremes of the support of μ_P.*

One of the nice corollary of the representation of the limit law in terms of free variables is that the eigenvalues are asymptotically connected in the sense that :

Corollary 2.7 *For any polynomial P in $\mathbf{U}, \mathbf{X}$, free unitaries and semi-circle respectively, the law μ_P has a connected support.*

Indeed, the C^* algebra generated by free semi-circle is projection less according to [PV82, GS09], which will not be the case if the support was disconnected. Indeed, otherwise the projection onto one of the connected component S of the support could be written $A = \int_\Gamma (z-P)^{-1} dz$ where Γ is a contour around S whose interior does not intersect the rest of the support. Since A belongs to the C^* algebra of P (as a continuous function of P) and therefore of a finite number of free semi-circle law (as $(\mathbf{X}, \mathbf{U})$ are in the C^*-algebra of free semi-circle variables), the conclusion follows.

2.3 R-transform and S-transform

We specialize the previous result to the case of the sum and the multiplication of free matrices.

Corollary 2.8 *Let D^N and $\tilde{D}^N$ be sequences of uniformly bounded self-adjoint matrices so that the empirical measure of their eigenvalues converge to μ and $\tilde{\mu}$ respectively. Let U^N be an independent unitary matrix following the Haar measure. Then*

1. *The spectral distribution of $D^N + U_N \tilde{D}^N U_N^*$ converges weakly almost surely to $\mu \boxplus \tilde{\mu}$ as N goes to infinity.*

2. Assume that D^N is nonnegative. Then, the spectral distribution of $(D^N)^{\frac{1}{2}} U_N \tilde{D}^N U_N^* (D^N)^{\frac{1}{2}}$ converges weakly almost surely to $\mu \boxtimes \tilde{\mu}$ as N goes to infinity.

It turns out that even though freeness looks much more complicated than independence, it still allows to make some computations, such as for instance computing $\mu \boxplus \tilde{\mu}$ or $\mu \boxtimes \tilde{\mu}$. This is done by characterizing a generating function of the moments of this law which is suitable for the problem under consideration. We shall in the following only consider these generating functions as formal series to simplify.

Let for a probability μ, $G_\mu(z) = \int (z-x)^{-1} d\mu(x)$ and let $z_\mu(g)$ be its inverse (which exists by Lagrange inversion theorem) whereas we denote s_μ the inverse of $\sum_{n \geq 1} z^n \mu(x^n)$. We then put

$$R_\mu(g) = z_\mu(g) - g^{-1} \qquad S_\mu(s) = \frac{1+s}{s} s_\mu(s) \,.$$

R_μ and S_μ both characterize the measure μ as it characterize its moments. Then, we have

Theorem 2.9

$$R_{\mu \boxplus \tilde{\mu}} = R_\mu + R_{\tilde{\mu}}$$
$$S_{\mu \boxtimes \mu'} = S_\mu S_{\mu'}$$

This result is often proved by using cumulants and the combinatorics of non-crossing partitions, see [NS06]. We here follow the point of view of T. Tao [Tao12] which gives a formal and clear argument for the additive property of the R-transform, but apply it to the S-transform for a change. We shall be sketchy below. We write S_X and s_X for the S-transform of the law μ (resp. s_μ) if X has law μ. The point is to write s_X as the solution of

$$\int \frac{x s_X(z)}{1 - s_X(z) x} d\mu(x) = z$$

which is equivalent to the existence of a centered random variable E_X so that

$$\frac{X s_\mu(z)}{1 - X s_\mu(z)} = z + E_X \qquad a.s\,.$$

Note that in fact E_X is a nice function of X, and can be approximated by polynomials at list for $\Im z$ large enough. Some algebra reveals that this is equivalent to

$$S_X(z)X = \frac{1+\frac{1}{z}E_X}{1+\frac{1}{z+1}E_X}\,.$$

Doing the same for Y with law $\tilde{\mu}$ we deduce

$$S_Y(z)S_X(z)XY = \left(\frac{1+\frac{1}{z}E_X}{1+\frac{1}{z+1}E_X}\right)\left(\frac{1+\frac{1}{z}E_Y}{1+\frac{1}{z+1}E_Y}\right)\,.$$

the statement will follow if we see that there exists E_{XY} centered so that

$$A := \left(\frac{1+\frac{1}{z}E_X}{1+\frac{1}{z+1}E_X}\right)\left(\frac{1+\frac{1}{z}E_Y}{1+\frac{1}{z+1}E_Y}\right) = \frac{1+\frac{1}{z}E_{XY}}{1+\frac{1}{z+1}E_{XY}}$$

or in other words if we can check that

$$E_{XY} := z(z+1)\frac{A-1}{z+1-zA}$$

is centered. This is equivalent to

$$E_{XY} = \frac{1}{z(z+1)-E_XE_Y}(E_X+E_Y+\frac{2z+1}{z(z+1)}E_XE_Y)$$

But it is not hard to verify that freeness implies that for all $n \in \mathbb{N}$

$$\tau[E_X(E_XE_Y)^n] = 0 \quad \tau[(E_XE_Y)^nE_Y] = 0 \quad \tau[(XY)^n] = 0$$

which shows that at least if we expand formally the denominator, E_{XY} is centered.

3 The spectrum of non-normal matrices and the Brown measure

Let us remind that non-normal matrices do not commute with their adjoint. This can also be written through the decomposition of the matrix in its eigenbasis. $X = QDQ^{-1}$ where the entries of D are the eigenvalues of X and Q is the matrix of the linearly independent eigenvectors; X is non-normal when we can not take Q unitary.

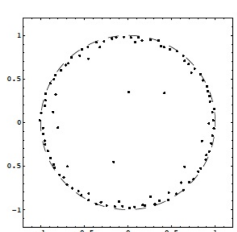
It turns out that the spectrum of non-normal matrices is not a smooth function of the entries. Indeed, the spectrum of non-normal matrices can be quite unstable, the addition of a matrix with very small entries being able to change it drastically, hence leading to the study of the pseudo-spectrum, see e.g. [TE05]. The typical example is given by the matrix

$$\Xi^{(N)} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

whose spectrum is reduced to the null eigenvalue. However, if one adds a very tiny matrix

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & N^{-10^6} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

the spectrum will be drastically modified as it is uniformly spread on the unit disc when N goes to infinity.



Such a phenomena can not happen for normal matrices, as can be seen thanks to the Hoffman-Wielandt inequality which shows that for two normal matrices A, B with complex-valued entries with eigenvalues $(\lambda_i(A))_{1\le i\le N}$ and $(\lambda_i(B))_{1\le i\le N}$ respectively, we have

$$\min_{\pi \text{ permutation}} \sum_{i=1}^{N} |\lambda_i(A) - \lambda_{\pi(i)}(B)|^2 \le \mathrm{Tr}(A-B)(A-B)^* \,.$$

This inequality shows that modifying the entries by a factor N^{-p} can not modify the spectrum of a normal matrix by an error greater than N^{-p+2}.

The point here is that even if the eigenvalues are smooth functions of the entries when they are distinct, as the roots of the characteristic polynomial, they are very close to each other so that in fact the Lipschitz constant blows up with N faster than polynomially if they are spaced like N^{-1}. Hence, adding a polynomially decaying matrix can affect the spectrum drastically.

This can be analytically understood by the Green formula. :If (λ_i^N) are the eigenvalues of X^N, for any smooth function ψ, we have

$$\begin{aligned} \sum_{i=1}^{N} \psi(\lambda_i^N) &= \frac{1}{2\pi} \int_{\mathbb{C}} \Delta\psi(z) \log |\prod_{i=1}^{N} (z-\lambda_i^N)| dz \\ &= \frac{1}{4\pi} \int_{\mathbb{C}} \Delta\psi(z) \left(\sum_{i=1}^{N} \log |z-\lambda_i^N|^2 \right) dz \\ &= \frac{N}{4\pi} \int_{\mathbb{C}} \Delta\psi(z) \int \log |x| dL_{(z-X^N)(z-X^N)^*}(x) dz \end{aligned}$$

where $L_{(z-X^N)(z-X^N)^*}$ denotes the spectral measure of the Hermitian matrix $(z-X^N)(z-X^N)^*$. As we have seen before, the convergence of the spectral measure $(z-X^N)(z-X^N)^*$ can be studied by moments methods and behaves smoothly with the entries of X^N. However, the logarithm is singular at the origin and can be seen to be exactly the source of the instability of the spectrum. The Brown measure is exactly defined as the measure we would obtain if we would "forget" this singularity. This is the idea originally proposed by Girko.

Namely, let us assume that the empirical distribution of $X^N, (X^N)^*$ converges to the non-commutative law τ and let L^z_X be the law of $(z-X)(z-X^*)$ under τ, that is the probability measure on the real line so that for each k

$$\int x^k dL^z_X(x) = \tau\left(((z-X)(z-X)^*)^k\right)$$

Then we define the Brown measure μ_X of X as the measure on $\mathbb{C}$ so that for all smooth functions ψ

$$\int \psi(z) d\mu_X(z) := \frac{1}{4\pi} \int_{\mathbb{C}} \Delta\psi(z) \int \log|x| dL^z_X(x) dz$$

Even though the Brown measure may not characterize the asymptotics of the spectrum of the matrices X^N, it will describe those of a small randomization of them. Indeed, Sniady [Śni02] proved the following

Theorem 3.1 *Let X^N be a sequence of $N \times N$ matrices so that the joint empirical distribution of $(X^N, (X^N)^*)$ converges. Let G_N be the non-normal matrix with i.i.d centered Gaussian entries with variance N^{-1}. Then, there exists ε_N going to zero as N goes to infinity so that the spectral measure $L_{X^N+\varepsilon_N G_N}$ converges towards the Brown measure of X.*

A similar result was obtained by Haagerup by smoothing via Cauchy random matrices.

The size of ε_N depends on the matrices X^N; in fact it can be taken to go to zero polynomially in the example $X^N = \Xi^N$, or for any matrices for which adding a polynomially small matrix (but eventually deterministic) is enough to obtain the convergence towards the Brown measure [GZW12]. However, some matrices require stronger regularization.

In the classical models of random non-normal matrices such as the Ginibre matrices which correspond to matrices with independent equidistributed entries, it turns out that the spectral measure converges towards the Brown measure, that is the uniform law on the disc as soon as the entries have (slightly more than) a finite second moment [Bai97, TV08].

In some cases, the Brown measure can be explicitly computed by using free probability tools. This is the case for instance for $X^N = U_N D_N$ with U_N following the Haar measure on the unitary group and D_N a diagonal matrix with real entries where this computation was done by Haagerup and Larsen [HL00]. Bitting the singularity of the logarithm (in particular thanks to recent results on the smallest singular value of $X^N + U_N$ obtained by M. Rudelson and R. Vershynin [RV12]), one can obtain [GKZ11]

Theorem 3.2 *Assume that D_N is uniformly bounded, with converging spectral measure and with Cauchy-Stieljes transform uniformly bounded on $\mathbb{C}^+$. Then, the spectral measure of $U_N D_N$ converges almost surely towards the Brown measure μ_{UD}. Moreover, the Brown measure μ_{UD} is rotation invariant and*

$$\mu_{UD}(B(0, f(t))) = t$$

where $f(t) = 1/\sqrt{S_{D^2}(t-1)}$ with S_{D^2} the S-transform of D^2. Furthermore, the support of μ_{UD} is an annulus

$$\mathrm{supp}(\mu_{UD}) = \{re^{i\theta}, r \in [(\mu_D(x^{-2}))^{-\frac{1}{2}}, (\mu_D(x^2))^{\frac{1}{2}}], \theta \in [0, 2\pi[\}$$

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