THE HOMOTOPY GROUPS OF THE $L_2$-LOCALIZED TODA-SMITH SPECTRUM $V(1)$ AT THE PRIME 3

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Abstract. In this paper, we try to compute the homotopy groups of the $L_2$-localized Toda-Smith spectrum $V(1)$ at the prime 3 by using the Adams-Novikov spectral sequence, and have almost done so. This computation involves non-trivial differentials $d_5$ and $d_9$ of the Adams-Novikov spectral sequence, different from the case $p > 3$. We also determine the homotopy groups of some $L_2$-localized finite spectra relating to $V(1)$. We further show some of the non-trivial differentials on elements relating so-called $\beta$-elements in the Adams-Novikov spectral sequence for $\pi_*(S^0)$.

Introduction

Let $L_n$ be the Bousfield localization functor from the category of spectra to itself with respect to $E(n)$ [19], and $V(k)$ denote the Toda-Smith spectrum [26] with $BP_*$-homology $BP_*/(p, v_1, \cdots, v_n)$, at each prime number $p$. Here $BP$ and $E(n)$ denote the Brown-Peterson and the Johnson-Wilson spectra with coefficient rings $BP_* = \mathbb{Z}[v_1, v_2, \cdots]$ and $E(n)_* = \mathbb{Z}[v_1, \cdots, v_n, v_n^{-1}]$ with $|v_n| = 2(p^n - 1)$. The Toda-Smith spectrum $V(k)$ is known to exist for $k < 4$ if and only if $p > 2k$ [26], [20], and note that $V(-1) = S^0$ and $V(0)$ is the mod $p$ Moore spectrum.

Determination of the homotopy groups of the $L_n$-localized sphere spectrum is one of the key problems to understanding the category of $L_n$-localized spectra. It is well known that $\pi_*(L_0S^0) = \mathbb{Q}$. The homotopy groups $\pi_*(L_1S^0)$ are determined by Ravenel [19] and $\pi_*(L_2S^0)$ is determined for $p > 3$ by Yabe and the author [24]. Thus the next case will be $\pi_*(L_2S^0)$ at the prime 3. At the prime $p > 3$, the homotopy groups $\pi_*(L_2S^0)$ are obtained from $\pi_*(L_2V(1))$ by the $v_1$- and $p$-Bockstein spectral sequences. Furthermore, at $p > 3$, the homotopy groups $\pi_*(L_2S^0)$ are isomorphic to the $E_2$-term of the Adams-Novikov spectral sequence, and they can be determined purely algebraically. Besides, $\pi_*(L_2V(1))$ is obtained from the cohomology of the Morava stabilizer algebra $S_2$, which is computed by Ravenel (cf. [20]). Different from the case $p > 3$, the homotopy groups at the prime 3 are not isomorphic to the $E_2$-term of the Adams-Novikov spectral sequence, and besides the $E_2$-term of the Adams-Novikov spectral sequence for computing $\pi_*(L_2V(1))$ is expressed by the language of cohomology of groups [4] (cf. [3], [28]). Actually, Henn noticed a mistake in [20, Th. 6.3.23] and gave the correction of it in [4] from the viewpoint of cohomology of groups. In this paper, we provide another verification of this result using the cobar complex in Section 5.

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The homotopy groups \( \pi_\ast(L_n V(n-1)) \) are computed by Ravenel [20] for \( n \leq 3 \) and \( p > n + 1 \), in which case the Adams-Novikov spectral sequence collapses. So the next case in this sense is \( p = n + 1 \), whose case involves non-trivial Adams-Novikov differentials. When \( n = 1 \), this is a celebrated result of Mahowald (cf. [6],[7],[8]). We study the case \( n = 2 \) here in this point of view.

In order to state our results, we introduce the notations:

\[
K(2)_\ast = \mathbb{Z}/3[v_2, v_3^{-1}], \quad K = \mathbb{Z}/3[v_2, v_2^9],
\]

\[
V = \mathbb{Z}/3\{v_2^j \mid j = 0, 1, 5\}, \quad \overline{V} = \mathbb{Z}/3\{v_2^j \mid j = 2, 3, 4, 6, 7, 8\},
\]

\[
P = \mathbb{Z}/3[b_{10}] \quad \text{and} \quad P_n = P/(b_{10}^n).
\]

Our main result is the following shown in Section 10.

**Theorem A.** The homotopy groups \( \pi_\ast(L_2 V(1)) \) at the prime 3 are isomorphic to the tensor product of the exterior algebra \( \Lambda(\zeta_2) \) and the direct sum of \( K \)-modules

\[
P_2 \otimes V \otimes K\{a_{-21}, a_{-24}'\} \oplus P_3 \otimes V \otimes K\{a_{51}, a_{38}'\} \\
\oplus P_4 \otimes V \otimes K\{a_{82}, a_{-3}'\} \oplus P_5 \otimes V \otimes K\{a_0, a_0'\}
\]

for the generators \( \zeta_2 \in \pi_{-1}(L_2 V(1)), a_i \in \pi_i(L_2 V(1)) \) and \( a_i' \in \pi_{4k+1}(L_2 V(1)) \).

Here, \( k \) is a fixed integer of \( \{0, 1, 2\} \).

Unfortunately, the integer \( k \in \{0, 1, 2\} \) stays undetermined but this theorem shows the size of the homotopy groups. The homotopy groups are computed by the Adams-Novikov spectral sequence, and are just the \( E_{10} \)-term (see Theorem 10.6) for them, in fact, \( E_{10}^s = 0 \) if \( s > 12 \).

On the way computing it, we determine the homotopy groups \( \pi_\ast(L_2 X \wedge V(1)) \) in Section 4 for the 8-skeleton \( X \) of \( BP \) with \( BP_\ast(X) = BP_\ast \{1, a, b\} \), where \( |a| = 4 \) and \( |b| = 8 \).

**Theorem B.** The homotopy groups \( \pi_\ast(L_2 X \wedge V(1)) \) are isomorphic to the tensor product of the exterior algebra \( \Lambda(\zeta_2) \) and the \( K(2)_\ast \)-module generated by \( b_0, b_5, b_6, b_{10}, b_{11} \) and \( b_{15} \) for the generators \( b_i \in \pi_i(L_2 X \wedge V(1)) \).

This is used to determine the \( E_2 \)-term of the Adams-Novikov spectral sequence for computing \( \pi_\ast(L_2 V(1)) \) (see Theorem 5.8). To show the relation \( d_\ast(\zeta_2) \in \zeta_2 E_1^\ast(L_2 V(1)) \) of the differential of the Adams-Novikov spectral sequence \( \{E_1^\ast(L_2 V(1))\} \) for \( L_2 V(1) \) (see Lemma 6.9), we compute the homotopy groups of \( L_2 V_1 \) for

\[
V_i = V(1) \cup_{\beta_i} \Sigma^{10i+1} V(1)
\]

in Section 6:

**Theorem C.** The homotopy groups \( \pi_\ast(L_2 V_1) \) are isomorphic to the tensor product of the exterior algebra \( \Lambda(\zeta_2) \) and the \( K(2)_\ast \)-module generated by \( a_0, a_3, a_6, a_{11}, a_{13}, a_{21}, a_{34} \) and \( a_{40} \) for the generators \( a_i \in \pi_i(L_2 V_1) \).

In the same way, we determine \( \pi_\ast(L_2 V_2) \). Furthermore, we compute \( \pi_\ast(L_2 V_3) \) to determine the Adams-Novikov differential \( d_5 \) on \( L_2 V(1) \) in Section 9.

**Theorem D.** The homotopy groups \( \pi_\ast(L_2 V_2) \) and \( \pi_\ast(L_2 V_3) \) are given as follows:

\[
\pi_\ast(L_2 V_2) = \pi_\ast(L_2 V_1) \otimes \Lambda(\beta_1), \quad \text{and}
\]

\[
\pi_\ast(L_2 V_3) = \Lambda(\zeta_2) \otimes K \otimes \{P_5 \otimes V \otimes F_1 \oplus \Lambda(g_3) \otimes \overline{V} \otimes (P_2 \otimes F_2 \oplus P_3 \otimes F_3)\}.
\]
Here \( xg_3 \in \pi_*(L_2V_3) \) denotes an element such that \( j_*(xg_3) = x \) for the projection \( j : V_3 \to \Sigma^{31}V(1) \), and

\[
F_1 = \mathbb{Z}/3\{a_0, a_{51}, a_{-21}, a_8, a'_9, a'_6, a'_{-69}, a'_{38}, a'_{-24}\}, \\
F_2 = \mathbb{Z}/3\{a_{-21}, a'_{-21}\} \quad \text{and} \quad F_3 = \mathbb{Z}/3\{a_{51}, a'_3\},
\]

for the elements \( a_i \) and \( a'_i \) in Theorem A.

Pemmaraju [16] recently shows that there exists a self map \( B : \Sigma^{144}V(1) \to V(1) \) with \( BP_*(B) = v_2^9 \). But here we show that \( \pi_*(L_2V(1)) \) is a \( \mathbb{Z}/3[v_2^9, v_2^{-9}] \)-module in a different way from his, using the homotopy element \( \beta_{6/3} \in \pi_*(S^0) \). Furthermore, his result shows the existence of \( \beta \)-elements \( \beta_t \) with \( t \equiv 0, 1, 2, 3, 5, 6 \ (9) \) (see Section 2 for the definition of the \( \beta \)-elements). As a corollary of Theorem A, we have

**Theorem E.** The \( \beta \)-element \( \beta_t \) exists in \( \pi_*(L_2S^0) \) if \( t \equiv 0, 1, 5 \ (9) \).

On the non-existence, we obtain, in the last section, the following

**Theorem F.** In the homotopy groups \( \pi_*(S^0) \), the \( \beta \)-element \( \beta_t \) does not exist if \( t \equiv 4, 7, 8 \ (9) \).

Note that for the case \( p > 3 \), the \( \beta \)-element \( \beta_t \) exists in \( \pi_*(S^0) \) for any \( t > 0 \). More generally, the \( \beta \)-elements \( \beta_{pt/p} \) and \( \beta_{p^i t/p, 2} \) are also shown to exist for \( p > 3 \) by Oka (cf. [20]). See Section 10 for the definition of these \( \beta \)-elements. At the prime 3, we show a generalization of Ravenel’s odd primary Kervaire invariant theorem:

**Theorem G.** The \( \beta \)-elements \( \beta_{9t+3/3}, \beta_{9s/3, 2} \) and \( \beta_{9s/3} \) do not exist in the homotopy groups \( \pi_*(S^0) \) for \( t \geq 1, s \neq 0 \ (3) \) and \( i > 1 \).

Recall the theorem of Hopkins-Gross that the Brown-Comenetz dual \( L_nF \) of a finite spectrum \( F \) with \( K(n)_*(F) \neq 0 \) and \( K(n-1)_*(F) = 0 \) is homotopic to the Spanier-Whitehead dual \( L_nDF \) of \( F \) up to suspension if \( p1_F \sim 0 \) and \( p \) is large enough to satisfy the inequality \( 2p - 2 \geq \max\{n^2, 2n + 2\} \). Our case \( n = 2 \) and \( p = 3 \) does not satisfy the condition, but it seems to hold true for our case. Here we can prove it for \( F = V_1 \) in Section 6 by using Theorem C.

**Theorem H.** \( \Sigma^{-39}L_2V_1 \simeq L_2V_1 \).

As is seen in the chart of Theorem 9.1, this kind of duality seems to hold even at \( p = 3 \). If we know the duality of this kind for \( L_2V(1) \) previously, the computation will be much easier.

This paper is organized as follows:
1. Some properties of the Adams-Novikov spectral sequence.
2. Some homotopy elements in \( \pi_*(V(1)) \)
3. Ravenel’s spectral sequence
4. The homotopy groups \( \pi_*(L_2X \wedge V(1)) \)
5. The Adams-Novikov \( E_2 \)-term for \( L_2V(1) \)
6. The homotopy groups of \( L_2V_1 \)
7. \( \mathbb{Z}/3[v_2^9, v_2^{-9}] \)-module structure
8. The Adams-Novikov differential on \( L_2V(1) \)
9. The homotopy groups of \( L_2V_2 \) and \( L_2V_3 \)
10. The \( E_{10} \)-term of the Adams-Novikov spectral sequence for \( L_2V \)
11. The non-existence of \( \beta \)-elements
The author would like to thank Don Davis for carefully reading the first draft of this paper, and not only kindly pointing out gaps in the proof of a lemma which is here Lemma 9.7, but also suggesting the author to prove that \( v_2 \) detects a homotopy element which first was quoted from Pemmaraju’s result.

1. Some properties of the Adams-Novikov spectral sequence

Throughout this paper, we consider everything localized at the prime 3. By \( BP \), we denote the Brown-Peterson spectrum whose coefficient ring and self homology are polynomial algebras

\[
BP_* = \mathbb{Z}_3[v_1, v_2, \cdots] \quad \text{and} \quad BP_*(BP) = BP_*[t_1, t_2, \cdots]
\]

over the generators \( v_i \) and \( t_i \) with degree \(|v_i| = 2(3^i - 1) = |t_i| \) for \( i > 0 \). This gives rise to a Hopf algebroid (cf. [1]). Consider the \( BP_*(BP) \)-comodule algebra \( E(2)_* = \mathbb{Z}_3[v_1, v_2, v_2^{-1}] \) whose \( BP_* \)-action is given by sending \( v_i \) to \( v_i \) for \( i \leq 2 \) and to 0 otherwise. Then \( E(2)_*(-) = E(2)_* \otimes_{BP_*} BP_*(-) \) is a homology theory by the Landweber exact functor theorem, and we denote \( E(2) \) for the spectrum representing the theory. It is a ring spectrum and yields the Hopf algebroid \( (A, \Gamma) = (E(2)_*, E(2)_*(E(2))) \) with

\[
\Gamma = E(2)_*(E(2)) = E(2)_*[t_1, t_2, \cdots]/(u_i : i > 2).
\]

Here \( u_i \) denotes the image of \( v_i \) under the homomorphism \( BP_* \eta_R \to E(2)_*[t_1, t_2, \cdots] \) for the right unit \( \eta_R \) of the Hopf algebroid \( BP_*(BP) \). Since \( \Gamma \) is flat over \( A \), the category of \( \Gamma \)-comodules has enough injectives, and \( \text{Ext}_\Gamma(A, -) \) is defined to be a derived functor of \( \text{Hom}_\Gamma(A, -) \).

Recall [2] the construction of the Adams-Novikov spectral sequence. Let \( E \) be a ring spectrum with the unit map \( i : S^0 \to E \), and \( E \) the cofiber of \( i \). Then we have the exact couple

\[
\begin{array}{c}
E^s \wedge G \xrightarrow{k^{s+1}} E^{s+1} \wedge G \\
i \wedge 1 \\
\pi \wedge E^s \wedge G
\end{array}
\]

for a spectrum \( G \). Consider the cofiber \( E_s \) of \( k^s : E^s \to \Sigma^s S^0 \), and we have another exact couple

\[
\begin{array}{c}
E_s \wedge G \xrightarrow{k_{s+1}} E_{s+1} \wedge G \\
i_s \\
E \wedge E^s \wedge G
\end{array}
\]

Applying the homology functor \( \pi_*(-) \) to this exact couple, we obtain the spectral sequence abutting to \( \pi_*(G) \). This spectral sequence is called a generalized Adams spectral sequence based on \( E \). This exact couple also gives rise to a finite spectral sequence converging to the homotopy groups \( \pi_*(E^0 \wedge G) \) with the same \( E_1 \)-term as the generalized Adams spectral sequence. Let \( H \) be a spectrum, and \( E_*(H) \) denote the \( E_* \)-term of these spectral sequences abutting or converging to \( \pi_*(H) \). The following is an easy consequence of the definition.
Lemma 1.2. For any integers \( n, r > 0 \), \( E_r^s(\overline{E}_n \wedge G) = 0 \) if \( s \geq n \), and \( E_r^s(G \wedge \overline{E}_n) = E_r^s(G) \) if \( s \leq n - r \).

Let

\[
X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X
\]

be a cofiber sequence with \( E_*(h) = 0 \). Then we have short exact sequences of \( E_1 \)-terms which yield a long exact sequence

\[
\cdots \rightarrow E^s_2(X) \xrightarrow{f_*} E^s_2(Y) \xrightarrow{g_*} E^s_2(Z) \xrightarrow{\partial} E^{s+1}_2(X) \rightarrow \cdots
\]

(1.3)

of \( E_2 \)-terms, where \( \partial \) denotes the connecting homomorphism. In the same way as [10, Th. 4.1], we see the following

Lemma 1.4. Suppose that the differential \( d_r : E^\ast_r(W) \rightarrow E^{\ast+r}_r(W) \) is trivial if \( r < n \) for \( W = X, Y, Z \). If \( g_\ast(y) \) is a non-trivial permanent cycle for \( y \in E^n_\ast(Y) = E_2(Y) \), then

\[
d_n(y) = f_\ast(h \circ z) \in E^{s+n}_n(Z) = E^{s+n}_2(Z),
\]

where \( z \) denotes a homotopy element representing \( g_\ast(y) \), and \( \pi \) denotes the projection of a homotopy element \( x \) of \( \pi_\ast(X) \) into \( E^{s+n}_2(X) \).

Take now \( E = E(2) \) (resp. \( BP \)). Then the spectral sequence converges to \( \pi_\ast(L^2G) \) (resp. \( \pi_\ast(G) \)) if \( G \) is connective (cf. [19]), and we call it the Adams-Novikov spectral sequence. Here \( L_2 \) denotes the Bousfield localization functor with respect to \( E(2) \). Let \( E^r_\ast(L^2G) \) (resp. \( E^r_\ast(G) \)) denote the \( E_r \)-term of the Adams-Novikov spectral sequence and it is known that

\[
E^r_2(L^2G) = \text{Ext}^{r,t}_{E^2}(A, E(2)_\ast(G))
\]

(1.4)

and

\[
E^r_2(G) = \text{Ext}^{r,t}_{BP\ast}(BP_\ast, BP_\ast(G)).
\]

Now recall the construction of the Toda-Smith spectrum \( V(1) \). Let \( M \) denote the cofiber of \( 3 \in \pi_0(S^0) = \mathbb{Z} \) and we have a cofiber sequence

\[
S^0 \xrightarrow{3} S^0 \xrightarrow{i} M \xrightarrow{j} S^1.
\]

(1.5)

Now the Toda-Smith spectrum \( V(1) \) is defined to be the cofiber of the Adams map \( \alpha \in [M, M]_4 \) which is characterized by \( BP_\ast(\alpha) = v_1 \). Thus we have the cofiber sequence

\[
\Sigma^4 M \xrightarrow{\alpha} M \xrightarrow{i_1} V(1) \xrightarrow{j_1} \Sigma^5 M.
\]

(1.6)

Since \( V(1) \) is an \( S^0 \)- and \( M \)-module spectrum (cf. [27]), it is well known that

\[
\text{For } x \in E^r_\ast(S) \text{ and } y \in E^r_{s,t}(L^2V(1)),
\]

\[
d_r(xy) = d_r(x)y + (-1)^{t-s}xd_r(y) \quad (\text{cf. [20, Th. 2.3.3]}),
\]

(1.7)

where \( S = S^0 \) or \( M \).

The Bockstein operation on \( V(1) \) is defined by the composition \( i_1j_1 \) of maps in (1.6), which induces the usual Bockstein operation \( \delta \) on \( E^2_2(L^2V(1)) \) given by

\[
\delta = i_1 \partial_1
\]

(1.8)

for the connecting homomorphism \( \partial_1 : E^2_2(L^2V(1)) \rightarrow E^{s+1}_2(L^2M) \).
Lemma 1.9. Let $F$ denote a finite spectrum such that $E(2)_*(F)$ is free over $E(2)_*$, and $x$ and $y$, elements of $E_2^1(L_2V(1) \wedge F)$. Suppose that $x$ and $y$ are permanent cycles such that $d_r(xy) = 0$ for $r < 5$, and that $\partial(xy) \neq 0$. Then

$$d_5(xy) = i_1\omega(x)\delta(y) \in E_2^2(L_2V(1) \wedge F),$$

where $\omega(x) \in E_2^2(L_2M \wedge F)$ denotes the projection of the Toda bracket $\langle \alpha, j_1(x), \alpha \rangle$ for the homotopy element $\tilde{x}$ detected by $x$.

Proof. The proof is almost identical to that of [10, Th. 4.2]. Notice that $\partial(xy)$ equals the Massey product $\langle \partial(x), v_1, \partial(y) \rangle$, and that $v_1$ detects the homotopy element $\alpha$. We denote a homotopy element $j_1\tilde{x}$ detected by $\partial(x)$. Then $\partial(xy)$ detects a homotopy element given by the Toda bracket $\langle j_1\tilde{x}, \alpha, j_1\tilde{y} \rangle$. By Lemma 1.4, we compute

$$d_5(xy) = i_1\alpha(j_1\tilde{x}, \alpha, j_1\tilde{y})$$
$$= i_1\alpha(j_1\tilde{x}, \alpha, j_1\tilde{y})$$
$$= i_1\alpha(j_1\tilde{x}, \alpha, j_1\tilde{y}).$$

q.e.d.

2. SOME HOMOTOPY ELEMENTS IN $\pi_*(V(1))$

Throughout this section we consider the Adams-Novikov spectral sequence $\{E^r_*(X)\}$ for computing $\pi_*(X)$. First we review the definition of $\alpha$- and $\beta$-elements. Let $\partial : E_2^2(M) \to E_2^{1+1}(S^0)$ and $\partial_1 : E_2^2(V(1)) \to E_2^{1+1}(M)$ be the connecting homomorphisms given in (1.3) from the cofiber sequences (1.5) and (1.6), respectively. Recall [20, Cor. 4.3.21] Landweber’s formula $\eta_T(v_n) \equiv v_n + v_{n-1}t_1^{3n-1} - v_{n-1}t_1 \mod (3, v_1, \cdots, v_{n-2})$. Then, we see that $v_1^t \in E_2^0(M)$ and $v_2^t \in E_2^0(V(1))$ for $t > 0$. Now, $\alpha_t$, $\beta'_t$ and $\beta_t$ are defined by

$$\alpha_t = \partial(v_1^t) \in E_2^1(S^0),$$
$$\beta'_t = \partial_1(v_2^t) \in E_2^1(M) \quad \text{and} \quad \beta_t = \partial \partial_1(v_2^t) \in E_2^2(S^0).$$

In [20], we find that $\alpha_t$ for $t > 0$ and $\beta_t$ for $t = 1, 2, 3, 5, 6$ are permanent as well as $\beta'_t$ for $t = 1, 2, 3, 5, 6$. Send these elements by the induced maps from the inclusions $i_1 : S^0 \to V(1)$ and $i_1 : M \to V(1)$, and we obtain the same named elements in $E_2^1(V(1))$. These are represented by elements of the cobar complex as follows (cf. [14, Lemma 5.4], [20]):

$$\alpha_t = h_{10}, \quad \beta'_t = tv_2^{-1}h_{11},$$
$$\beta_{3k+1} = v_2^{3k}b_{10} \quad \text{and} \quad \beta_{3k+2} = v_2^{3k+1}h_{11} \zeta_2,$$

for non-negative integers $k, t$ with $t > 0$. In a similar way, we also have another $\beta$-element $\beta_{3t/3}$, which is represented as follows (cf. [14, Lemma 5.4]):

$$\beta_{3t/3} = sv_2^{3t-3}b_{11} \quad \text{and} \quad \beta_{9s/3} = sv_2^{9s-3}b_{11}.$$

Here the homology classes $h_{11}$, $b_{11}$ and $\zeta_2$ are represented by cocycle of the cobar complex $\Omega^2_2(E(2)_*)$ as follows:

$$h_{11} = [t^3], \quad b_{11} = [(t_1 \otimes t_1^* + t_1^2 \otimes t_1)^3], \quad \text{and} \quad \zeta_2 = [v_2^{-1}t_2 + v_2^{-3}(t_2^3 - t_1^2)].$$

For the Adams-Novikov spectral sequence for $\pi_*(L_2V(1))$, we have the following
Lemma 2.4. In the $E_2$-term $E_2^*(L_2V(1))$ of the Adams-Novikov spectral sequence,
\[ h_{10}, \quad v_2^3 h_{10}, \quad h_{11}, \quad v_2^2 h_{11}, \quad v_2^3 h_{11}, \quad b_{10}, \quad v_2^2 h_{11} \zeta_2 \quad \text{and} \quad v_2^3 h_{11} \]
are all permanent cycles. Therefore, if $x$ denotes one of the elements, then
\[ d_*(yx) = d_*(y)x \]
for any $y \in E_2^*(L_2V(1))$.

Proof. By [20, Table A3.4], we see that $\alpha_1, \beta_1, \beta_2, \beta_5$ and $\beta_6/3$ are essential homotopy elements of $\pi_* (S^0)$, and these are pulled back to $\pi_* (M)$ and denoted by $\alpha'_1, \beta'_1, \beta'_2$ and $\beta'_6/3$, since they have order 3. By (2.2) and (2.3), we see that these homotopy elements represent the elements of the $E_2$-term for $V(1)$ as follows:
\[ \alpha_1 = h_{10}, \quad \beta_1 = b_{10}, \quad \beta_2 = -v_2 h_{11} \zeta_2, \quad \beta_5 = -v_2^2 h_{11} \zeta_2, \quad \beta_6/3 = -v_2^3 h_{11}, \quad \alpha'_1 = v_1, \quad \beta'_1 = h_{11}, \quad \beta'_2 = -v_2 h_{11}, \quad \beta'_5 = -v_2^2 h_{11}, \quad \text{and} \quad \beta'_6/3 = -v_2^3 h_{10}. \]
Here $\beta'_6/3$ is also read off from [14, (4.5)] using the relation $t_1^0 \equiv v_2^2 t_1 \mod (3,v_1)$ obtained from $u_3 = 0$ in (1.1). This implies the former half. The latter half follows from (1.7), immediately. q.e.d.

From here on, we consider the homotopy groups of $V(1)$, the unlocalized one. If $v_2^s$ of the $E_2$-term survives to $\beta^{(i)}$ of $\pi_*(V(1))$, then $\beta_i$ in the $E_2$-term detects the same named element
\[ (\beta_1 = jj_1(\beta^{(i)})) \in \pi_{16i-6}(S^0), \]
for $j$ and $j_1$ in (1.5) and (1.6) by the Geometric Boundary theorem (cf. [20, Th. 2.3.4]. As to $\beta^{(i)}$, we have the following

Theorem 2.6. If $t = 0, 1$ or 5, then $v_2^s$ survives to $E_\infty$-term, that is, it detects a homotopy element $\beta^{(i)} \in \pi_*(V(1))$.

Proof. For $t = 0$, we put $\beta^{(0)} = 1 \in \pi_0(V(1))$. Since $E_{2t}^*(V(1)) = 0$ unless $t = 0$ mod 4 and $E_{2t+2, s+17}^* = 0$ if $s > 1$, we see that $d_{2t}(v_2) = 0$ in $E_{2t}^*(V(1))$, and that $v_2$ detects the homotopy element $\beta^{(1)}$.

Now we consider $v_2^5$. We will show that the homotopy element $\beta_5 \in \pi_{74}(S^0)$ is pulled back to $\pi_{80}(V(1))$ by the map $j j_1 : \Sigma^0 V(1) \rightarrow S^0$. If this is shown, then we take $\beta^{(5)} = (jj_1)^{-1}(\beta_5)$ by virtue of (2.5). The table of Ravenel's book [20, Table A.3.4] shows
\[ \pi_{75}(S^0) = \mathbb{Z}/3\{\alpha_{19}\} \oplus \mathbb{Z}/9\{x_{75}\} \quad \text{and} \quad \pi_{74}(S^0) = \mathbb{Z}/3\{\beta_3\}; \quad \text{and} \quad \pi_{79}(S^0) = \mathbb{Z}/3\{\alpha_{20}\} \quad \text{and} \quad \pi_{78}(S^0) = \mathbb{Z}/3\{\beta_1 x_{68}\}. \]
Consider the long exact sequence
\[ \cdots \xrightarrow{3} \pi_{75}(S^0) \xrightarrow{\iota} \pi_{75}(M) \xrightarrow{j_*} \pi_{74}(S^0) \xrightarrow{3} \cdots \]
associated to the cofiber sequence (1.5), and we obtain
\[ \pi_{75}(M) = \mathbb{Z}/3\{\iota_*(\alpha_{19}), \iota_*(x_{75}), \beta_1 x_{68}\} \]
for $\beta_5$ such that $j_*(\beta_5) = \beta_5$. In the same way, we obtain
\[ \pi_{79}(M) = \mathbb{Z}/3\{\iota_*(\alpha_{20}), \beta_1 x_{68}\} \]
for $\tilde{\beta}_1$ such that $j_*(\tilde{\beta}_1) = \beta_1$. Note that $\alpha i x_{75} \in \pi_{79}(M)$ by degree reason, and we compute

$$j_*(\alpha i x_{75}) = a_1 x_{75} = a_1 (a_1, a_1, x_{68}) = (a_1, a_1, a_1) x_{68} = \beta_1 x_{68} = j_*(\tilde{\beta}_1 x_{68}).$$

Therefore, we put

$$\alpha i x_{75} = \tilde{\beta}_1 x_{68} + ki_*(\alpha_{20})$$

for some $k \in \mathbb{Z}/3$. Now consider the Adams-Novikov filtration, which is defined as follows:

$$\text{filt } x = n \text{ for } x \in \pi_*(X) \text{ if and only if } x \text{ is detected by a non-trivial element of } E_2^n(X).$$

Since $i_*(\alpha_{20}) = \alpha_{20}$ in the $E_2$-term, we see that $\text{filt } i_*(\alpha_{20}) = 1$. We also read off $\text{filt } \alpha i x_{75} = \text{filt } \tilde{\beta}_1 x_{68} = 5$ from [20]. Therefore, $k = 0$ and so

$$\alpha i x_{75} = \tilde{\beta}_1 x_{68}.$$  

We note that $\tilde{\beta}_5$ is detected by the element $\partial_1(v_2^5)$ in the $E_2$-term. We also see that $v_1 \partial_1(v_2^5) = 0$ in the $E_2$-term by the definition of $\partial_1$, which means $\text{filt } \alpha \tilde{\beta}_5 > 1$. Furthermore, $\text{filt } i_*(\alpha_{20}) = 1 = \text{filt } \tilde{\beta}_1$. Therefore we read off the relation

$$\alpha \tilde{\beta}_3 = k \tilde{\beta}_1 x_{68}$$

for some $k \in \mathbb{Z}/3$, from the homotopy group $\pi_{79}(M)$. Now define

$$\beta'_5 = \tilde{\beta}_5 - ki_*(x_{75}) \in \pi_{75}(M).$$

Then, $j_*(\beta'_5) = \beta_5$, and (2.8) and (2.9) show

$$\alpha \beta'_5 = 0.$$ 

This implies the existence of an element $\beta^{(5)} \in \pi_{80}(V(1))$ such that $j_1*(\beta^{(5)}) = \beta'_5$, as desired.

3. Ravenel’s spectral sequence

Let $X$, $Y$ and $Y'$ be the spectra defined by

$$X = S^0 \cup_{\alpha_1} e^4 \cup_{-\alpha_1} e^8, \quad Y = S^0 \cup_{\alpha_1} e^4 \quad \text{and} \quad Y' = S^0 \cup_{-\alpha_1} e^4.$$

Note that $X$ and $Y$ are the 8-skeleton and the 4-skeleton of $BP$, respectively. Then we have cofiber sequences

$$Y' \xrightarrow{i} X \xrightarrow{j} S^8 \xrightarrow{k} \Sigma Y' \quad \text{and} \quad S^0 \xrightarrow{i} X \xrightarrow{j'} \Sigma^4 Y' \xrightarrow{k'} S^4.$$

Here $i : Y' \to X$ is defined to be the composition $Y' \xrightarrow{i} Y \subset X$ for $i$ fitting into the commutative diagram

$$\begin{array}{ccc}
S^3 & \xrightarrow{-\alpha_1} & S^0 \\
\downarrow & & \downarrow \\
S^3 & \xrightarrow{\alpha_1} & S^0
\end{array}$$

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Applying the homotopy functor $\pi_*(-)$, we have two long exact sequences
\[
\cdots \to \pi_*(G \wedge Y') \xrightarrow{i_*} \pi_*(G \wedge X) \xrightarrow{j_*} \pi_{*-4}(G) \to \cdots, \quad \text{and}
\cdots \to \pi_*(G) \xrightarrow{i_*} \pi_*(G \wedge X) \xrightarrow{j_*} \pi_{*-4}(G \wedge Y') \to \cdots
\]
for any spectrum $G$. These associate the spectral sequence
\[
E_1^{x,t} = (\wedge(\alpha_1) \otimes \mathbb{Z}[[\beta_1]] \otimes \pi_*(G \wedge X))^{x,t} \Rightarrow \pi_{t-x}(G)
\]
with bidegree $||\alpha_1|| = (1,4)$, $||\beta_1|| = (2,12)$ and $||x|| = (0,t)$ for $x \in \pi_t(G \wedge X)$.

Taking $E(2) \wedge G$ for $G$ above, we have the short exact sequences
\[
0 \to E(2)_*(G \wedge Y') \xrightarrow{i_*} E(2)_*(G \wedge X) \xrightarrow{j_*} E(2)_{*-4}(G) \to 0, \quad \text{and}
0 \to E(2)_*(G) \xrightarrow{i_*} E(2)_*(G \wedge X) \xrightarrow{j_*} E(2)_{*-4}(G \wedge Y') \to 0,
\]
for $E_*(X)$ and $E_*(Y')$ are $E_*(G)$-free over generators with degree 0 mod 4. Applying $\text{Ext}_*^{x,t}(A,-)$ to these, we obtain an exact couple which gives us a spectral sequence
\[
E_1^{x,t} = (A \otimes \mathbb{Z}[[b_{10}]] \otimes \text{Ext}_*^{x,t}(A, E(2)_*(G \wedge X)))^{x,t} \Rightarrow \text{Ext}_*^{x,t}(A, E(2)_*(G)).
\]
Here the bidegrees are given as follows:
\[
||b_{10}|| = (1,4), \quad ||b_{10}|| = (2,12) \quad \text{and}
||x|| = (s,t) \quad \text{for} \quad x \in \text{Ext}_*^{x,t}(A, E(2)_*(G \wedge X)).
\]
Note that the classes $h_{10}$ and $b_{10}$ converge to the generators $\alpha_1$ and $\beta_1$, respectively, of the $E_2$-term of the Adams-Novikov spectral sequence for spheres (cf. (2.2)).

4. The Homotopy Groups $\pi_*(L_2X \wedge V(1))$

Recall (1.6) the definition of the Toda-Smith spectrum $V(1)$. We use the following notation as is done in [15]:
\[
V = V(1) \quad \text{and} \quad VX = V \wedge X = V(1) \wedge X,
\]
and obtain that
\[
E(2)_*(V) = K(2)_* = \mathbb{Z}/3[v_2, v_2^{-1}] \quad \text{and} \quad E(2)_*(VX) = K(2)_*[t_1]/(t_1^3),
\]
as a $\Gamma$-comodule. Then the $E_2$-term of the Adams-Novikov spectral sequence converging to the homotopy groups $\pi_*(L_2VX)$ is
\[
\text{Ext}_*^{x,t}(A, K(2)_*[t_1]/(t_1^3)).
\]
In order to compute this, we use a change of rings theorem. For this purpose, we introduce Hopf algebras over $\mathbb{Z}/3$
\[
S_* = \mathbb{Z}/3[t_i]/(t_1 - t_0^i), \quad T_* = \mathbb{Z}/3[t_i]/(t_0^i),
\]
\[
S(2)_* = \mathbb{Z}/3[t_1, t_2, \cdots]/(t_i - t_0^i : i \geq 1) \quad \text{and}
\]
\[
S(2,2)_* = \mathbb{Z}/3[t_1, t_2, t_3, \cdots]/(t_i - t_0^i : i > 1),
\]
whose structure maps are read off from that of $\Gamma$, or that of $BP_*(BP)$ (cf. [20, Chap. 4]). For example, for $\Delta : S(2)_* \to S(2)_* \otimes S(2)_*$,
\[
\Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1,
\]
\[
\Delta(t_2) = t_2 \otimes 1 + t_1 \otimes t_1^3 + 1 \otimes t_2 \quad \text{and}
\]
\[
\Delta(t_3) = t_3 \otimes 1 + t_1 \otimes t_2^3 + t_2 \otimes t_1^3 + 1 \otimes t_3 + b_{11}.
\]
Here,

\[
b_{11} = t^4_1 \otimes t^6_1 + t^6_1 \otimes t^3_1.
\]

We then have the extension of Hopf algebras over \( \mathbb{Z}/3 \)

\[
S_* \longrightarrow S(2)_* \longrightarrow S(2,2).
\]

by which we have the Cartan-Eilenberg spectral sequence (cf. [20, A1.3.14])

\[
E^s_{2t} = \text{Ext}^s_{S_*}([\mathbb{Z}/3, \text{Ext}^t_{S(2)_*}([\mathbb{Z}/3, \mathbb{Z}/3(t_1)/(t^3_1)]))
\]

\[
\Rightarrow \text{Ext}^{s+t}_{S(2)_*}([\mathbb{Z}/3, \mathbb{Z}/3(t_1)/(t^3_1)]).
\]

Here the \( E_2 \)-term is computed as follows:

\[
E^s_{2t} = \text{Ext}^s_{S_*}([\mathbb{Z}/3, \text{Ext}^t_{S(2)_*}([\mathbb{Z}/3, \mathbb{Z}/3(t_1)/(t^3_1)]))
\]

\[
= \text{Ext}^s_{S_*}([\mathbb{Z}/3, \text{Ext}^t_{S(2)_*}([\mathbb{Z}/3, \mathbb{Z}/3)])]
\]

by the change of rings theorem [20, A1.3.12], since \( \mathbb{Z}/3(t_1)/(t^3_1) = S_* \square_{T_*} \mathbb{Z}/3 \) seen by the structure (4.1).

**Lemma 4.3.** ([20, Th. 6.3.7])

\[
\text{Ext}^{s}_{S(2,2)_*}(\mathbb{Z}/3, \mathbb{Z}/3) = \Lambda(h_{20}, h_{21}, h_{30}, h_{31}).
\]

Here \( h_{ij} \) is represented by \( t^3_i \) of the cobar complex \( \Omega^2_{S(2,2)_*}, \mathbb{Z}/3 = S_*(2,2)_* \).

**Theorem 4.4.** The \( E_2 \)-term of the Adams-Novikov spectral sequence for \( \pi_*(L_2VX) \) is given by

\[
E^s_{2t} = \Lambda(\zeta_2) \otimes K(2)_s \{1, h_{11}, h_{20}, \xi, \varphi, h_{20}\xi\}.
\]

Here \( \zeta_2, \xi \) and \( \varphi \) are represented by \( h_{20} + h_{21}, h_{11}, h_{30} \) and \( (h_{20} - h_{21})h_{31} \), and so the bidegrees of the generators are:

\[
||\zeta_2|| = (1,0), \quad ||h_{11}|| = (1,12), \quad ||h_{20}|| = (1,0),
\]

\[
||\xi|| = (2,8) \quad \text{and} \quad ||\varphi|| = (2,12).
\]

**Proof.** By the structure map (4.1), the equation (4.2) turns into

\[
E^s_{2t} = \text{Ext}^s_{T_*}([\mathbb{Z}/3, \mathbb{Z}/3] \otimes \text{Ext}^s_{S(2,2)_*}(\mathbb{Z}/3, \mathbb{Z}/3),
\]

and it is well known that

\[
\text{Ext}^s_{T_*}([\mathbb{Z}/3, \mathbb{Z}/3] = \Lambda(h_{11}) \otimes \mathbb{Z}/3[b_{11}].
\]

The differential \( d_2 \) of the Cartan-Eilenberg spectral sequence is computed to be

\[
d_2(h_{2i}) = 0 \quad \text{for} \quad i = 0,1; \quad \text{and}
\]

\[
d_2(h_{30}) = -b_{11} \quad \text{and} \quad d_2(h_{31}) = -h_{11}(h_{20} - h_{21})
\]

by (4.1). Now note that

\[
E_2^s(L_2VX) = K(2)_s \otimes \text{Ext}^s_{S(2,2)_*}(\mathbb{Z}/3, \mathbb{Z}/3(t_1)/(t^3_1)),
\]

and we obtain the theorem. q.e.d.

Since \( E_2^s(L_2VX) = 0 \) unless \( t \equiv 0 \mod 4 \), the Adams-Novikov differential \( d_r = 0 \) for \( r \leq 4 \). The theorem shows that \( E_2^s(L_2VX) = 0 \) for \( s > 4 \), which shows \( d_2 = 0 \) for \( r > 4 \). Therefore, the spectral sequence collapses to the \( E_2 \)-term. Since \( V(1) \) is an \( M \)-module spectrum [27], there is no extension problem. Hence the \( E_2 \)-term gives the homotopy groups of \( L_2VX \).

**Corollary 4.5.** \( \pi_*(L_2VX) = \Lambda(\zeta_2) \otimes K(2)_s \{1, h_{11}, h_{20}, \xi, \varphi, h_{20}\xi\} \).
5. The Adams-Novikov $E_2$-term for $L_2V(1)$

For the sake of convenience, we use the abbreviations:

\[
\begin{align*}
\text{Ext}^* & = \text{Ext}^*_{\mathcal{K}}(A, M) \quad \text{for a } \Gamma\text{-comodule } M, \\
V & = V(1), \quad VX = V(1) \wedge X, \\
VV & = V(1) \wedge Y, \quad VY' = V(1) \wedge Y', \\
K(2)_* & = E(2)_*(V) = \mathbb{Z}/3[v_1v_2^{-1}], \\
KY & = E(2)_*(VV) = K(2)_* \oplus K(2)_* \{a\}, \\
KY' & = E(2)_*(VY') = K(2)_* \oplus K(2)_* \{a'\} \quad \text{and} \\
KX & = E(2)_*(VX) = K(2)_* \oplus K(2)_* \{a\} \oplus K(2)_* \{b\}.
\end{align*}
\]

Here $X$, $Y$ and $Y'$ are the spectra in (3.1). The $\Gamma$-comodule structure $\psi$ (resp. $\psi'$) of $KX$ and $KY$ (resp. $KY'$) satisfies

\[
\psi(a) = a + t_1 \quad \text{and} \quad \psi(b) = b - at_1 + t_1^2. \quad \text{(resp. } \psi'(a') = a' - t_1).\]

**Lemma 5.1.** The map $i : Y' \to Y$ in (3.2) induces an isomorphism $i_* : KY' \to KY$ of comodules such that $i_*(1) = 1$ and $i_*(a') = -a$.

Taking $G$ to be $L_2V$, the short exact sequences of (3.4) induce the long ones

\[
\begin{align*}
\text{Ext}^* & \xrightarrow{\partial} \text{Ext}^*K(2)_* \xrightarrow{j} \text{Ext}^*K(2)_* \xrightarrow{\partial'} \text{Ext}^*K(2)_*, \\
\text{Ext}^* & \xrightarrow{\partial} \text{Ext}^*K(2)_* \xrightarrow{j} \text{Ext}^*K(2)_* \xrightarrow{\partial'} \text{Ext}^*K(2)_*,
\end{align*}
\]

with the connecting homomorphisms

\[
\partial : \text{Ext}^*K(2)_* \longrightarrow \text{Ext}^{*+1}KY' \quad \text{and} \quad \partial' : \text{Ext}^{*}KY' \longrightarrow \text{Ext}^{*+1}K(2)_*.
\]

**Lemma 5.3.** As a $\lambda(\zeta_2)$-module map, the maps $j_*$ and $j'_*$ of (5.2) send elements as follows:

\[
\begin{align*}
&j_*(1) = 0, \quad j_*(h_{11}) = 0, \quad j_*(h_{20}) = 0, \\
&j_*(\xi) = 0, \quad j_*(\varphi) = 0, \quad j_*(h_{20}) = 0; \quad \text{and} \\
&j'_*(1) = 0, \quad j'_*(h_{11}) = 0, \quad j'_*(h_{20}) = h_{11}, \\
&j'_*(\xi) = 0, \quad j'_*(\varphi) = \xi, \quad j'_*(h_{20}) = 0.
\end{align*}
\]

**Proof.** In this proof, we omit $v_2$'s, because they just play a role adjusting the internal degrees. For a cocycle $x + ax_1 + bx_2$ of the cobar complex $\Omega^*KX$,

\[
\begin{align*}
&j_*(x + ax_1 + bx_2) = x_2 \quad \text{and} \quad j'_*(x + ax_1 + bx_2) = x_1 + ax_2,
\end{align*}
\]

where $x, x_1, x_2 \in \Omega^*K(2)$. We notice that $\xi$ and $\varphi$ are represented by Massey products

\[
\xi = -\langle h_{11}, h_{11}, h_{20} - h_{21} \rangle \quad \text{and} \quad \varphi = \langle h_{20} - h_{21}, h_{11}, h_{20} - h_{21} \rangle.
\]

Here the product of $KX$ used to define these Massey products is defined by

\[
(x + ax_1 + bx_2)(y + ay_1 + by_2) = xy + a(xy_1 + xy_1) + b(x_2y + x_1y_1 + xy_2).
\]

Now we see easily that the cocycles $h_{11}$, $h_{20}$ and $h_{20} - h_{21}$ are represented as follows:

\[
\begin{align*}
h_{11} &= [t_1^3], \quad h_{20} = [t_2 + at_1^2], \quad h_{20} - h_{21} = [A + at_1^2],
\end{align*}
\]

in which $A = t_2 - t_1^2 + t_1^4$. These show the lemma for $1, h_{11}$ and $h_{20}$.

On the other hand, $\xi$ is known to exist in $\Omega^2K(2)_*$ (cf. [20]), and so $j_*(\xi)$ and $j'_*(\xi)$ are both 0, and so is for $h_{20}\xi$. 


Since \( h_{11}(h_{20} - h_{21}) = 0 \), there exists a cochain \( z \) such that \( d(z) = (A + at_1^2) \otimes t_2^3 \).
Using \( z, \xi \) and \( \varphi \) are represented by \( \xi = [-t_1^2 \otimes z + \cdots] \) and \( \varphi = [-A \otimes z - a t_1^2 \otimes z + \cdots] \).
Therefore, we obtain \( j_1(\varphi) = 0 \) and \( j'_1(\varphi) = \xi \).

**Lemma 5.4.** (i) For an element \( x \in \text{Ext}^* K(2)_* \),
\[
\partial(x) = t_1^2 \otimes x + a' t_1 \otimes x.
\]
(ii) For an element \( y = y_1 + a' y_2 \in \text{Ext}^* KY' \),
\[
\partial'(y) = t_1 \otimes y_1 + t_1^2 \otimes y_2.
\]

**Proof.** (i) The equation \( \partial(x) = w \) follows from the relation \( d(bx) = i_*(w) \) in the cobar complex \( \Omega^*_A KX \) by the definition of \( \partial \). Now compute
\[
d(bx) = -at_1 \otimes x + t_1^2 \otimes x,
\]
since \( x \) is a cocycle.

(ii) Write \( y = y_1 + a' y_2 \). By definition, \( \partial'(y) \) follows from the computation \( d(ay_1 + by_2) \). Since \( y \) is a cocycle, \( d(y_1) = t_1 \otimes y_2 \) and \( d(y_2) = 0 \). Then,
\[
d(ay_1 + by_2) = t_1 \otimes y_1 + at_1 \otimes y_2 - at_1 \otimes y_2 + t_1^2 \otimes y_2.
\]
q.e.d.

First we compute the \( E_2 \)-terms of Ravenel’s spectral sequence (3.5) with \( G = V \):

**Lemma 5.5.** The \( E_2 \)-terms are isomorphic, as a \( K(2)_*[b_{10}] \)-module, to the \( K(2)_*[b_{10}] \)-module
\[
\Lambda(\zeta_2) \otimes K(2)_*[b_{10}]\{1, h_{10}, h_{11}, \xi, b_{11}, \psi_0, \psi_1, b_{11} \xi\}.
\]
Here \( \psi_0, \psi_1 \) and \( b_{11} \) are represented by \( h_{10} \varphi, h_{20} \xi \) and \( h_{10} h_{20} \), respectively, and their bidegrees are: \( ||\psi_0|| = (3, 16), ||\psi_1|| = (3, 24) \) and \( ||b_{11}|| = (2, 36) \).

**Proof.** We have the only non-trivial differentials:
\[
d_1(h_{20}) = h_{10} h_{11} \quad \text{and} \quad d_1(\varphi) = h_{10} \xi,
\]
which is seen by Lemmas 5.3 and 5.4. These show the lemma. q.e.d.

**Lemma 5.7.** In the spectral sequence (3.5) with \( G = V \),
\[
d_r = 0
\]
for \( r > 1 \).

**Proof.** Due to [20, p.239], \( h_{10}, h_{11}, b_{10}, b_{11}, \xi \) and \( b_{11} \xi \) are the cocycles of \( \Omega^*_A K(2)_* \).
Furthermore, \( \psi_k \) is represented by the Massey product \( \langle h_{1k}, h_{10}, \xi \rangle \) for \( k = 0, 1 \), and so these are also cocycles of the complex. This means that every element in the \( E_2 \)-term is a permanent cycle of the spectral sequence (3.5) as desired. q.e.d.

Therefore, \( E_2 = E_\infty \) in the spectral sequence of (3.5). So Lemma 5.5 is restated as follows:

**Theorem 5.8.** The \( E_2 \)-term of the Adams-Novikov spectral sequence for computing \( \pi_*(L_2 V(1)) \) is isomorphic, as a \( P \)-module, to the \( P \)-module
\[
P \otimes K(2)_*, F \otimes \Lambda(\zeta_2).
\]
Here,
\[
P = K(2)_*[b_{10}] \quad \text{and} \quad F = K(2)_*[1, h_{10}, h_{11}, b_{11}, \xi, \psi_0, \psi_1, b_{11} \xi]
\]
and \( \psi_k = \langle h_{1k}, h_{10}, \xi \rangle \) (Massey product).
Proposition 5.9. We have the following multiplicative relations:

\[ h_{10}b_{11} = 0, \quad h_{10}\xi = 0 \quad \text{and} \quad h_{11}\xi = 0. \]

Moreover, we have

\[
\begin{align*}
    v_2^2b_{10} & = -b_{11}b_{10} = h_{11}b_{11}, \\
    v_2h_{11}b_{10} & = -h_{10}b_{11}, \\
    b_{11}\xi & = v_2h_{11}\psi = v_2h_{11}b_0, \\
    b_{10}\psi & = -v_2^{-1}h_{11}\psi, \\
    v_2^3b_{10} & = -b_{11}^2, \\
    v_2h_{11}\psi & = -v_2^{-1}b_{11}\psi \\
    b_{10}\psi & = v_2^2b_{11}\psi
\end{align*}
\]

read off from the relation of Massey products.

Proof. The triviality \( h_{10}b_{11} = 0 \) and \( h_{10}\xi = 0 \) follow from the relation (5.6). \( h_{11}\xi = 0 \) follows from Theorem 4.4. In fact, by degree reason, Theorem 5.8 indicates \( h_{11}\xi = xv_2^{-1}b_{11}\xi \) for some \( x \in \mathbb{Z}/3 \). The edge homomorphism sends this to the same one in \( E_2^*(L_2VX) \). In \( E_2^*(L_2VX) \), \( h_{11}\xi = 0 \), and so \( x = 0 \).

For the second half, we have:

\[
\begin{align*}
    \langle h_{10}, h_{11}, h_{10} \rangle & = -b_{10}, \\
    \langle h_{11}, h_{10}, h_{10} \rangle & = \langle h_{10}, h_{11}, h_{10} \rangle = v_2^{-1}b_{11} + v_2b_{10}\xi_2 \quad \text{and} \\
    \langle h_{11}, h_{11}, h_{10} \rangle & = v_2b_{10} + v_2^2\xi_2h_{11}.
\end{align*}
\]

Therefore, we see that

\[
\begin{align*}
    v_2^3b_{10} & = v_2(h_{11}, h_{11}, h_{10})h_{10} = h_{10}\psi_0 = h_{10}(h_{10}, h_{10}, \xi) \\
    v_2h_{11}b_{10} & = v_2b_{11}(h_{11}, h_{10}, h_{10}) = v_2^{-1}h_{11}\psi_1 = v_2^{-1}(h_{11}, h_{10}, \xi) \\
    v_2h_{11}\psi_1 & = v_2(h_{11}, h_{11}, h_{10})h_{10} = b_{10}\psi_0 = b_{10}(h_{10}, h_{10}, \xi) \\
    v_2h_{11}\psi_0 & = v_2(h_{11}, h_{10}, h_{10}) = b_{10}\psi_1 = b_{10}(h_{10}, h_{10}, \xi) \\
    v_2b_{10} & = v_2(h_{11}, h_{11}, h_{10}) = b_{10}\psi_0 = b_{10}(h_{10}, h_{10}, \xi)
\end{align*}
\]

These have no indeterminacy by degree reason.

For the other relation,

\[
v_2^3b_{10} = -v_2^3b_{10}(h_{10}, h_{10}, h_{10}) \subset -v_2^3b_{10}(h_{10}, h_{10}, h_{10})
\]

which contains \(-b_{11}^2\) with indeterminacy \( \mathbb{Z}/3\{v_2^3b_{10}b_{11}\xi_2\} \). So put \( v_2^3b_{10} = -b_{11}^2 + kv_2^2h_{11}b_{10}\xi_2 \) and we will show that \( k = 0 \). Consider a \( \Gamma \)-comodule \( A_i = A_1/(3, v_2^i) \) for \( i > 0 \). Then a short exact sequence \( 0 \to A_1 \to A_2 \to A_1 \to 0 \), which associates a long exact sequence

\[
\cdots \to \Ext^s(A_1) \xrightarrow{v_1} \Ext^s(A_2) \xrightarrow{v_2} \Ext^s(A_1) \xrightarrow{v_3} \cdots
\]

By observing the Massey products, we see

\[
\begin{align*}
    v_2^3b_{10} & = -b_{11}^2 + kv_2^3h_{11}b_{10}\xi_2 + lv_2^2h_{10}\psi_0
\end{align*}
\]

in \( \Ext^4(A_2) \). We prepare the following lemma which is proved later.

Lemma 5.12. \( v_2^3h_{11}b_{11}b_{10}\xi_2 \) represents a non-trivial element of \( \Ext^5(A_2) \).

Apply \( h_{10} \in \Ext^1(A) \) to (5.11) to see \( v_2^3h_{10}b_{10}^2 = -h_{10}b_{11}^2 + kv_2^3h_{10}h_{11}b_{11}\xi_2 \).

We also have \( v_2^2h_{10}b_{10} = h_{11}b_{11} \) and \( v_2h_{11}b_{10} = -h_{10}b_{11} \) in \( \Ext^3(A_2) \) by the same fashion as in \( \Ext^3(A_1) \), because here there is no indeterminacy, either. Thus we
have $v_2^3b_{12}b_{10} = -h_{10}b_{11}$ and so $kv_2^3h_{10}b_{11} = 0$. Now Lemma 5.12 implies $k = 0$ as desired.

Proof of Lemma 5.12. In the cobar complex, $v_2^3h_{10}b_{11}$ is homologous to $-v_1v_2^3b_{10}b_{11}$ by $d(v_2^2b_2b_{11})$. Therefore, it is a $v_1$-image of $-v_2^2b_{10}b_{11}$. Furthermore, Theorem 5.8 implies $\text{Ext}^4(A_1)$.

There exists a pairing $\nu$.

Pick out $1$.

Furthermore, we see that $[\nu_1]$.

Lemmma 6.2.

Proof. Recall [27, Th. 6.11] the track groups of $V$:

$[V, V]_{-7} = 0$, $[V, V]_{-6} = \mathbb{Z}/3\{\delta_0\}$, $[V, V]_{-5} = \mathbb{Z}/3\{\delta_0\}$, $[V, V]_{-4} = \mathbb{Z}/3\{\beta_0\}$, $[V, V]_{-3} = \mathbb{Z}/3\{\beta_0\}$ and $[V, V]_{-2} = \mathbb{Z}/3\{\beta_0\}$.

Observing the exact sequences associating to the cofiber sequence $\Sigma^1V \to V$.

Proposition 6.3. There exists a pairing $\nu : V \to V$ which is an extension of the identity $1 : V \to V$.

Proof. Let $M$ denote the mod 3 Moore spectrum as before, and we have the splitting $M \wedge V = V \vee \Sigma V$.

since $M \wedge V = V \vee \Sigma V$. This yields two maps $m : M \wedge V \to V$ and $d : \Sigma V \to M \wedge V$

as a projection and an inclusion. Consider the exact sequence $[V \wedge V, V]_0 \to [M \wedge V, V]_0 \to [M \wedge V, V]_1$. 

q.e.d.
Here $[M \wedge V_1, V_1]_4 = [V_1, V_1]_5$ under the induced map from $d$ by Lemma 6.2. Then we see that $\alpha^*(m) = 0$, since it is the composition $\Sigma^5 V \xrightarrow{d} \Sigma^4 \wedge V \xrightarrow{\alpha^1} \wedge V \xrightarrow{m} V$, which is shown to be trivial in [23, Lemma 2.6]. Thus we have the desired map $\nu_1$ by pulling $m$ back under $i_1^*$.

**Corollary 6.4.** $V_1$ is a ring spectrum.

**Proof.** By Proposition 6.1, we have a splitting

$$V_1 \wedge V_1 = (V \wedge V_1) \vee \Sigma^{11} (V \wedge V_1),$$

which defines a map $j : V_1 \wedge V_1 \to V \wedge V_1$. Now the structure map is set to be a composition $\nu_1 j$.

Here by a ring spectrum, we mean a spectrum $X$ together with maps $\mu : X \wedge X \to X$ and $\eta : S^0 \to X$ such that the composition

$$X = S^0 \wedge X \xrightarrow{\eta \wedge X} X \wedge X \xrightarrow{\mu} X$$

is the identity.

We now consider the $E_2$-terms $E_2^2(\Sigma F_2 k)$ of the Adams-Novikov spectral sequence for computing $\pi_*(\Sigma F_2 k)$.

**Proposition 6.5.** We have a $K(2)_+\module$ isomorphism

$$E_2^2(\Sigma F_2 k) \cong P \otimes F \otimes \Lambda(\zeta_2, g_k)$$

for $k > 0$. Here $|g_k| = 10k + 1$, and $P$ and $F$ are those of Theorem 5.8.

**Proof.** By degree reason, one of $E_2^{2,k}(\Sigma F_2 k)$ and $E_2^{2,k-10k-1}(\Sigma F_2 k)$ is trivial. The cofiber sequence $\Sigma^{10k} V \xrightarrow{b_k} V \to V_k \to \Sigma^{10k+1} V$ gives rise to the split exact sequence of $E_2$-terms

$$0 \to E_2^{2,t}(\Sigma F_2 k) \to E_2^{2,t}(\Sigma F_2 k) \to E_2^{2,t-10k-1}(\Sigma F_2 k) \to 0.$$  

Now use Theorem 5.8 to get the proposition.

**Lemma 6.7.** For an element $x \in P \otimes F \otimes \Lambda(\zeta_2)$,

$$d_2(xg_1) = b_{10} x \in E_2^2(\Sigma F_2 k) \text{ and } d_4(xg_2) = b_{10}^2 x \in E_2^4(\Sigma F_2 k).$$

**Proof.** Let $xg_2 \in E_2^3(\Sigma F_2 k)$ for $k = 1, 2$, and consider the cofiber sequence

$$\Sigma^{10k} V \wedge E_{s+5} \xrightarrow{\beta_k} V \wedge E_{s+5} \xrightarrow{\iota} V_k \wedge E_{s+5} \xrightarrow{\jmath} E_{s+5} \wedge E_{s+5}.$$

Then it satisfies $E(2)_*(\Sigma^k) = 0$. Furthermore, we see that $j_*(xg_2) = x$ is a permanent cycle of the Adams-Novikov spectral sequence for $\pi_*(V \wedge E_{s+3})$, since $d_r(x) = 0$ for $r < 5$ by degree reason and for $r > 4$ by Lemma 1.2. By Lemma 1.4,

$$d_{2k}(xg_2) = i_*(\beta_k x) = i_*(b_{10}^k x) = b_{10}^k x \in E_2^{2k+2k}(V_k \wedge E_{s+5}),$$

since $\beta_1$ represents $b_{10}$. Since $L_2 V_k = \lim_s V_k \wedge E_s$, we have a map $L_2 V_k \to V_k \wedge E_{s+5}$ and so we see the lemma by the naturality of the differential.

**Theorem 6.8.** $\pi_*^e(\Sigma F) \cong F \otimes \Lambda(\zeta_2)$ as a $K(2)_+\module$. 

q.e.d.
Proof. The short exact sequence (6.6) gives rise to the long exact sequence
\[ \cdots \to E_2^s(L_2V) \xrightarrow{\delta} E_3^s+2(L_2V) \to E_3^s+2(L_2V_1) \to E_3^s+2(L_2V) \xrightarrow{\delta} \cdots, \]
since the $E_3^s$-term is a homology of the complex $(E_2^s, d_2)$. By the definition of the connecting homomorphism $\delta$, Lemma 6.7 implies
\[ \delta(x) = b_{10}x. \]
Therefore, we obtain
\[ E_3^s(L_2V_1) = F \otimes \Lambda(\zeta_2), \]
which is described in a chart as follows:

\begin{center}
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|c|}
\hline
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \mod 16 \hline
0 & $b_{11} \zeta_2$ & $b_{10} \zeta_2$ & $\xi_2$ & $s_1 \zeta_2$ & $b_{11} \xi$ & $s_1 \zeta$ & $b_0 \zeta_2$ & $s_0 \zeta_2$ & & \hline
1 & $s_1 \zeta$ & $s_1 \zeta$ & $s_1 \zeta$ & $s_1 \zeta$ & $s_1 \zeta$ & $s_1 \zeta$ & $s_1 \zeta$ & $s_1 \zeta$ & & \hline
2 & & & & & & & & & & \hline
3 & & & & & & & & & & \hline
\hline
\end{tabular}
\end{center}

This chart shows that $E_3^s(L_2V_1) = E_3^s(L_2V_1)$ and no extension problem arises. Therefore, $\pi_*(L_2V_1) = E_3^s(L_2V_1)$ as desired. q.e.d.

Using this theorem, we obtain one of the main lemmas for determining the homotopy groups of $L_2V(1)$:

**Lemma 6.9.** In the Adams-Novikov spectral sequence $\{E_2^s(L_2V)\}$ for computing $\pi_*(L_2V)$, let $x \in E_2^s(L_2V)$ denote an element belonging to the subquotient of $P \otimes F (V = V(1))$. Then,
\[ d_r(x \zeta_2) = y \zeta_2 \in E_r^s(L_2V) \]
for some $y \in E_2^s(L_2V)$ corresponding to $y \in P \otimes F \subset E_2^s(L_2V)$.

Proof. Theorem 6.8 certifies the existence of the homotopy element $\zeta \in \pi_*(L_2V_1)$ detected by $\zeta_2$. Consider the composition $z : V = V \cong S^0 \xrightarrow{1 \zeta} \Sigma V \wedge V_1 \xrightarrow{\sigma} \Sigma V_1$ and the cofiber sequence
\[ \Sigma^{-1}V \xrightarrow{\zeta} V_1 \xrightarrow{\sigma} C_z \xrightarrow{\jmath} V. \]
Then $E(2)_*(z) = 0$, since $E(2)_*(V) = K(2)_*$ and $E(2)_*(V_1) = K(2)_* \otimes \Lambda(g_1)$, and so we obtain the short exact sequence
\[ 0 \to K(2)_* \otimes \Lambda(g_1) \xrightarrow{i_z} E(2)_*(C_z) \xrightarrow{\jmath_*} K(2)_* \to 0. \]
This associates the long exact sequence of $E_2$-terms:
\[ \cdots \to P \otimes F \otimes \Lambda(\zeta_2) \xrightarrow{\zeta_2} P \otimes F \otimes \Lambda(\zeta_2, g_1) \xrightarrow{i_z} E_2^s(L_2C_z) \xrightarrow{\jmath_*} \cdots, \]
which gives us
\[ E_2^s(L_2C_z) = i_* (P \otimes F \oplus P \otimes F \otimes \Lambda(\zeta_2) g_1) \oplus j_*^{-1} (P \otimes F \zeta_2). \]
By Lemma 6.7, we compute
\[ d_2(i_* (x g_1)) = i_* d_2(x g_1) = i_* (b_{10}x) \in i_* (P \otimes F) \subset E_2^s(L_2C_z) \]
for $x \in P \otimes F$. Therefore,
\[ E_3^s(L_2C_z) = i_* (F \oplus P \otimes F \zeta_2 g_1) \oplus j_*^{-1} (P \otimes F \zeta_2). \]
For $x\xi_2 \in E'_v(L_2 V)$ survived from $P \otimes F\xi_2 \subset E'_2(L_2 V)$, we compute
$$d_r(x\xi_2) = d_r(j_*(x\xi_2)) = j_*(d_r(x\xi_2))$$
$$\in (\text{the subquotient of } j_*(E'^{s+r}_2(L_2 C_2)))$$
$$= (\text{the subquotient of } P \otimes F\xi_2) \subset E'^{s+r}_r(L_2 V). \text{ q.e.d.}$$

From here on in this section, we will consider the duality. Let $u \in \pi_*(L_2 V_1)$ denote the homotopy element detected by the element $b_{11}\xi_2 \in E_\infty(L_2 V_1)$. Then we have an element $u^* \in [V_1, I_2]_{-39} = \text{Hom}(\pi_{39}(L_2 V_1), Q/\mathbb{Z}_3)$ dual to $u$. Here $I_2$ denotes the Brown-Comenetz dual of $L_2 S^0$. Consider also a composition
$$\overline{u} : \Sigma^{-39} V_1 \wedge V_1 \xrightarrow{u} \Sigma^{-39} V_1 \xrightarrow{u^*} I_2.$$

Consider the adjoint
$$u^\#: \Sigma^{-39} V_1 \longrightarrow \text{Map}(V_1, I_2)$$
of this map $u$. Here $\text{Map}(G, I_2) = I_2 G$ is the Brown-Comenetz dual of $G$, which indicates $\pi_*(I_2 G) = \text{Hom}(\pi_*(G), Q/\mathbb{Z}_3)$. The computation of the homotopy groups, Theorem 6.8 and Proposition 5.9, show the following

Corollary 6.10. The adjoint map $L_2 u^\#$ is homotopy equivalent.

In fact, the chart in Theorem 6.8 is written as follows using the dual elements $x^*$ such that $xx^* = b_{11}\xi_2$:

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
\hline
10 & 11 & 12 & 13 & 14 & 15 & 16 & & \\
\hline
\end{array}$$

7. $\mathbb{Z}/3[v_2^9, v_3^9]$-module structure

In this section, we will show the following proposition which will give the homotopy groups $\pi_*(L_2 V(1))$ a $\mathbb{Z}/3[v_2^9, v_3^9]$-module structure.

Lemma 7.1. In the Adams-Novikov spectral sequence for computing $\pi_*(L_2 V(1))$, if $d_r(x)b_{10} = 0$, then $d_r(x) = 0$.

Proof. Consider the cofiber sequence
$$\Sigma^{10} V \xrightarrow{d_{11}} V \xrightarrow{i_{11}} V_1 \longrightarrow \Sigma^{11} V$$
for $V = V(1)$. As in (6.6), we have the short exact sequence
$$0 \longrightarrow E'^{s+t}_2(L_2 V) \xrightarrow{i_{s+t}} E'^{s+t}_2(L_2 V_1) \longrightarrow E'^{s+t-11}_2(L_2 V) \longrightarrow 0.$$ 

Since $d_2(xg_1) = b_{10}x$ in $E'^{s+t}_2(L_2 V_1)$ by Lemma 6.7, the short exact sequence yields the long one:
$$\cdots \longrightarrow E'^{s-2, t-12}_3(L_2 V) \xrightarrow{b_{10}} E'^{s, t}_3(L_2 V) \xrightarrow{i_{s+t}} E'^{s, t}_3(L_2 V_1) \longrightarrow E'^{s, t-11}_3(L_2 V) \longrightarrow \cdots.$$ 

Suppose that $d_r(x) \neq 0$. Then by Theorem 5.8, the equation $d_r(x)b_{10} = 0$ in $E'^{s}_r(L_2 V)$ implies that $d_r(x)b_{10}$ must be killed. Let $y$ denote the killer, and $i_{11}(y)$ is a permanent cycle by Theorem 6.8. Notice that any element in $E'_2(L_2 V)$ for $s > 5$...
is divided by $b_{10}$. By a diagram chasing with Lemma 6.7, we see that $xb_{10}$ is
a permanent cycle. Therefore, we have $i_1(xb_{10}) = i_1(y)$ in the homotopy
groups $\pi_*(L_2V_1)$. By Theorem 6.8, the Adams-Novikov filtration of $i_1(y)$ is
less than 6 if $i_1(y)$ survives to $E_2^*(L_2V_1)$. Since $y$ has a greater filtration
than that of $xb_{10}$, the difference is no less than 4 by degree reason. Therefore, $\text{filt}^A N(xb_{10}) \leq \text{filt}^A N i_1(y) - 4 < 6 - 4 = 2$, which
contradicts the fact that $\text{filt}^A N b_{10} = 2$. So suppose that $i_1(y)$ is killed. Then, $i_1(y) = zb_{10} = d_2(zg_1)$ for some $z$ by Lemma 6.7 and
Theorem 6.8. Then by a diagram chasing in the Adams-Novikov exact couple, we
see that $z = x$, which also contradicts our hypothesis. Therefore $d_r(x) = 0$ as
desired.

q.e.d.

The following chart may help to understand the above proof.

$$
\begin{array}{c|c|c|c}
\text{d}_r(x) & \to & \text{d}_r(x)b_{10} & \vdash \text{d}_s, \\
\downarrow \text{d}_r & & \downarrow \text{d}_s & \vdash \text{i}_1(y) \\
x & \Rightarrow & \text{xb}_{10} & \Rightarrow \text{i}_1(y) \\
\mathcal{V} & \Rightarrow \text{xb}_{10} & \Rightarrow \mathcal{V} & \Rightarrow \text{i}_1(y) \\
\end{array}
$$

**Proposition 7.2.** In the Adams-Novikov spectral sequence for computing
$\pi_*(L_2V(1))$,

$$
d_r(v_2^{9r}x) = v_2^{9r}d_r(x)
$$

for integers $t \geq 0$ and $r > 1$ and for a class $x$ of $E_2^*(V(1))$.

**Proof.** We proceed by induction on $t$. It is trivial for $t = 0$. Suppose that $d_r(v_2^{9r}x) = v_2^{9r}d_r(x)$. By Lemma 2.4, $v_2^3b_{11}$ is a permanent cycle, and so is $(v_2^3b_{11})^2 = v_2^5b_{11} = v_2^9b_{10}$. Therefore, we have

$$v_2^{9r}d_r(x)(v_2^3b_{10}) = d_r(v_2^{9r}x)(v_2^3b_{10}) = d_r(v_2^{9r+9}x)b_{10} = d_r(v_2^{9r+9}x)b_{10}
$$

by (1.7). Now apply Lemma 7.1 to see $d_r(v_2^{9r+9}x) = v_2^{9r+9}d_r(x)$. q.e.d.

8. The Adams-Novikov Differential on $L_2V(1)$

In this section, we compute the differential $d_5$ of the Adams-Novikov spectral
sequence for computing $\pi_*(L_2V)$, using the so-called Toda differential. Recall [25]
the Toda differential:

$$d_5(b_{11}) = \lambda h_{10}b_{10}^3 \text{ in } E_5^*(S^0) = E_2^*(S^0)
$$

for some non-zero element $\lambda \in \mathbb{Z}/3$. Consider the composition $f : S^0 \to V(1) \to L_2V(1)$ of the inclusion to the bottom cell and the localization map.

**Lemma 8.2.** In $E_5^*(L_2V)$,

$$f_*d_5(b_{11}^2) = \lambda v_2h_{11}b_{10}^4
$$

for the induced map $f_* : E_5^*(S^0) \to E_5^*(L_2V)$.

**Proof.** By the derivation property and (8.1),

$$d_5(b_{11}^2) = -\lambda h_{10}b_{10}^3b_{11},
$$

which is sent to $\lambda v_2h_{11}b_{10}^4$ by the map $f_*$, since $h_{10}b_{11} = -v_2h_{11}b_{10}$ by Proposition 5.9. q.e.d.
Lemma 8.3. If \( d_5(v^j_2) = xv^{j-2}_2h_{11}b_{10}^2 \) for some \( x \in \mathbb{Z}/3 \) in \( E^5_6(L_2V) \), then

\[
\begin{align*}
d_5(v^{j+3}_2) &= (x - \lambda)v^{j+1}_2h_{11}b_{10}^3 & \text{in } E^5_6(L_2V), \\
d_5(v^3_2h_{10}) &= 0 & \text{in } E^5_6(L_2V), \\
d_5(v_2^j b_{11}) &= 0 & \text{in } E^5_6(L_2V) \text{ and} \\
d_5(v^{j+3}_2b_{11}) &= xv^{j+3}_2h_{10}b_{10} & \text{in } E^7_5(L_2V).
\end{align*}
\]

Proof. We compute first:

\[
\begin{align*}
d_5(v^{j+3}_2b_{10}) &= -d_5(v_2^2b_{11}) & \text{by } b_1^2 = -v_2^3b_{10} \text{ in Proposition 5.9}, \\
&= -d_5(v_2^j b_{11}) - v_2^3d_5(b_{11}^2) & \text{by (1.7)}, \\
&= -xv^{j-2}_2h_{11}b_{10}^2 - \lambda v^{j+1}_2h_{11}b_{10}^4 & \text{by the hypothesis and Lemma 8.2}.
\end{align*}
\]

Since \( b_{10} \) acts monomorphically by Theorem 5.8, we obtain the first equation of the lemma. The other equations follow from Lemma 2.4, using the hypothesized equation, the derivation property and Proposition 5.9. \( \text{q.e.d.} \)

Now Theorem 2.6, Proposition 7.2 and Lemma 8.3 yield

Proposition 8.4. The differential \( d_5 : E^5_6(L_2V) \to E^5_{6+5}(L_2V) \) for \( v^j_2, v^3_2h_{10}, \) \( v^j_2b_{11} \) and \( v^2_2b_{11} \) is given as follows:

\[
\begin{align*}
d_5(v^j_2) &= \begin{cases} 0, & j \equiv 0, 1, 5 \mod 9, \\
-\lambda v^{j-2}_2h_{11}b_{10}^2, & j \equiv 3, 4, 8 \mod 9, \\
\lambda v^{j-2}_2h_{11}b_{10}^2, & j \equiv 2, 6, 7 \mod 9. \end{cases} \\
d_5(v^3_2h_{10}) &= 0, \\
d_5(v^j_2b_{11}) &= 0, \\
d_5(v^2_2b_{11}) &= \begin{cases} \lambda v^{j}_2h_{10}b_{11}b_{10}^2, & j \equiv 0, 1, 5 \mod 9, \\
0, & j \equiv 3, 4, 8 \mod 9, \\
-\lambda v^2_2h_{10}b_{11}b_{10}^2, & j \equiv 2, 6, 7 \mod 9. \end{cases}
\end{align*}
\]

Lemma 1.4 together with this proposition immediately implies the following corollary, since \( v_2 \) detects a homotopy element of \( \pi_*(L_2V) \).

Corollary 8.5. Let \( x \in E^5_6(L_2V) \) be a permanent cycle with \( \delta(v_2x) \neq 0 \). Then

\[
d_5(v_2x) = \lambda \delta(x)b_{10}^2 + \mu \delta(x)h_{11}\zeta_2b_{10} \quad (\mu \in \mathbb{Z}/3),
\]

where \( \delta \) denotes the Bockstein operator given in (1.8).

Proof. By the definition of \( \delta \), \( \delta(v_2) = h_{11} \) is seen by Landweber’s formula \( \eta_H(v_2) = v_2 + v_1 t_1^4 \mod (3, v_2^4) \). Since \( v_2 \) is a permanent cycle of the Adams-Novikov spectral sequence, Lemma 1.9 shows \( d_5(v_2x) = \iota_{1+}(\omega(v_2))b(x) \). We also have \( d_5(v_2^3) = \lambda h_{11}b_{10}^2 \) by Proposition 8.4. Comparing these, we obtain \( \iota_{1+}(\omega(v_2)) \equiv \lambda b_{10}^2 \mod \text{Ker } h_{11} \). Proposition 5.9 and Theorem 5.8 show that \( \text{Ker } h_{11} = \mathbb{Z}/3\{h_{11} \zeta_2 b_{10}\} \) at the degree \( |b_{10}^2| = 20 \). \( \text{q.e.d.} \)
Lemma 8.6. Suppose that \( d_5(v_2^j \psi_0) = xv_2^{j-3} \xi b_{11} b_{10}^2 \) for some \( x \in \mathbb{Z}/3 \) in \( E_5^6(L_2 V) \). Then,

\[
\begin{align*}
  d_5(v_2^{j+3} \psi_0) &= (x - \lambda)v_2^j \xi b_{11} b_{10}^2 \quad \text{in } E_5^6(L_2 V), \\
  d_5(v_2^j \xi) &= 0 \quad \text{in } E_5^6(L_2 V), \\
  d_5(v_2^{-1} b_{11} \xi) &= 0 \quad \text{in } E_5^6(L_2 V) \quad \text{and} \\
  d_5(v_2^{j+4} \psi_1) &= xv_2^{j+3} \xi b_{10}^4 \quad \text{in } E_5^6(L_2 V).
\end{align*}
\]

Proof. It follows from a direct computation using (1.7), Lemma 8.2 and Proposition 5.9 that

\[
\begin{align*}
  d_5(v_2^{j+3} \psi_0 b_{10}^2) &= -d_5(v_2^j \psi_0 b_{11}^2) \\
  &= -d_5(v_2^j \psi_0) b_{11}^2 - v_2^j \psi_0 d_5(b_{11}^2) \\
  &= -xv_2^{j-3} \xi b_{11} b_{10}^2 b_{11}^2 - \lambda v_2^{j+1} \psi_0 h_{11} b_{10}^2 \\
  &= xv_2^j \xi b_{11} b_{10}^2 + \lambda v_2^j \xi b_{11} b_{10}^4 \\
  &= (x - \lambda)v_2^j \xi b_{11} b_{10}^4.
\end{align*}
\]

This implies the first equation.

The second and third ones follow from the hypothesized equation, since \( h_{10} \psi_0 = -b_{10} \xi, h_{11} \psi_0 = v_2^{-3} b_{11} \xi \) and \( h_{1k} \xi = 0 \) for \( k = 0, 1 \) by Proposition 5.9.

The last one is also obtained from the similar computation:

\[
\begin{align*}
  d_5(v_2^{j+4} b_{10} \psi_1) &= -d_5(v_2^{j+3} b_{11} \psi_0) \\
  &= -d_5(v_2^j \psi_0) v_2^j b_{11} \quad \text{by Lemma 2.4} \\
  &= xv_2^{j+3} b_{10}^4,
\end{align*}
\]

using the relations in Proposition 5.9. q.e.d.

In the same way as above, we obtain the following:

Lemma 8.7. Suppose that \( d_5(v_2^j \zeta_2) = xv_2^{j-2} h_{11} b_{10}^2 \zeta_2 \) in \( E_5^6(L_2 V) \) and \( d_5(v_2^j \psi_0) = x'v_2^{j-3} \xi b_{11} b_{10}^2 \zeta_2 \) in \( E_5^6(L_2 V) \) for some \( x, x' \in \mathbb{Z}/3 \). Then,

a) \[
\begin{align*}
  d_5(v_2^{j+3} \zeta_2) &= (x - \lambda)v_2^{j+1} h_{11} b_{10}^2 \zeta_2 \quad \text{in } E_5^6(L_2 V), \\
  d_5(v_2^j h_{10} \zeta_2) &= 0 \quad \text{in } E_5^6(L_2 V), \\
  d_5(v_2^j h_{11} \zeta_2) &= 0 \quad \text{in } E_5^6(L_2 V) \quad \text{and} \\
  d_5(v_2^{j+3} b_{11} \zeta_2) &= xv_2^{j+3} h_{10} b_{10}^2 \zeta_2 \quad \text{in } E_5^6(L_2 V).
\end{align*}
\]

b) \[
\begin{align*}
  d_5(v_2^{j+3} \psi_0 \zeta_2) &= (x' - \lambda)v_2^j \xi b_{11} b_{10}^2 \zeta_2 \quad \text{in } E_5^6(L_2 V), \\
  d_5(v_2^j \xi \zeta_2) &= 0 \quad \text{in } E_5^6(L_2 V), \\
  d_5(v_2^{-1} b_{11} \xi \zeta_2) &= 0 \quad \text{in } E_5^6(L_2 V) \quad \text{and} \\
  d_5(v_2^{j+4} \psi_1 \zeta_2) &= x'v_2^{j+3} \xi b_{10}^2 \zeta_2 \quad \text{in } E_5^6(L_2 V).
\end{align*}
\]

Lemma 8.8. The hypothesis of the above lemma is satisfied. That is, \( d_5(v_2^j \zeta_2) = xv_2^{j-2} h_{11} b_{10}^2 \zeta_2 \) and \( d_5(v_2^j \psi_0 \zeta_2) = x'v_2^{j-3} \xi b_{11} b_{10}^2 \zeta_2 \) for \( j \in \mathbb{Z} \) and \( x, x' \in \mathbb{Z}/3 \).

Proof. By Theorem 5.8, we may put

\[
\begin{align*}
  d_5(v_2^j \zeta_2) &= xv_2^{j-2} h_{11} b_{10}^2 \zeta_2 + yv_2^{j-2} \psi_1 \zeta_2 b_{10} + zv_2^{j-2} b_{10}^3 \\
  d_5(v_2^j \psi_0 \zeta_2) &= x'v_2^{j-3} \xi b_{11} b_{10}^2 \zeta_2 + y'v_2^{j-3} \psi_1 \zeta_2 b_{10} + z'v_2^{j-3} b_{10}^3,
\end{align*}
\]

for some \( x, y, z \in \mathbb{Z}/3 \) by comparing degrees. By Lemma 2.4, \( h_{11} \) sends this to

\[
\begin{align*}
  d_5(v_2^j h_{11} \zeta_2) &= yv_2^{j-1} \xi \zeta_2 b_{10} + zv_2^{j-2} h_{11} b_{10}^3,
\end{align*}
\]
since $h_{11}\psi_1 = v_2 \xi b_{10}$ by Proposition 5.9. On the other hand, Proposition 8.4 says $d_5(v_2^{j-1}) = w v_2^{j-3} h_{11} b_{10}^2$ for some $w \in \mathbb{Z}/3$. The element $\beta_2 = v_2 h_{11} \zeta_2$ sends this to

$$d_5(v_2^j h_{11} \zeta_2) = 0$$

by (1.7). Therefore we have $y = 0 = z$ as desired.

For the other equations, it follows immediately from Theorem 5.8 and Lemma 6.9.

9. THE HOMOTOPY GROUPS OF $L_2 V_2$ AND $L_2 V_3$

In this section, we will determine the homotopy groups of $L_2 V_2$ and $L_2 V_3$ for $V_k = V \cup \beta_1 \Sigma^{10k} V$ ($V = V(1)$). Recall from Theorem 5.8 the $K(2)_{*}$-module

$$F = K(2)_{*} \{1, h_{10}, h_{11}, \xi, \psi_0, \psi_1, b_{11} \xi\}.$$

Our first result in this section is:

**Theorem 9.1.** $\pi_* (L_2 V_2) \cong F \otimes \Lambda (\zeta_2, b_{10})$ as a $K(2)_{*}$-module.

**Proof.** Consider the Adams-Novikov spectral sequence $\{E_*^r (L_2 V_2)\}$. By (6.6) and Lemma 6.7, we have the long exact sequence

$$(9.2) \quad \cdots \rightarrow E_*^5 (L_2 V) \xrightarrow{\delta} E_*^{5+4} (L_2 V) \xrightarrow{\delta} \cdots$$

with $\delta(x) = b_{10}^2 x$. Therefore,

$$E_*^5 (L_2 V_2) = F \otimes \Lambda (\zeta_2, b_{10}),$$

and we have the chart:

Here $x^*$ denotes an element such that $xx^* = b_{10} h_{11} \xi \zeta_2$. The element $x$ can be read off from Proposition 5.9. By this chart, $d_5(x) = 0$ unless $x \in F = K(2)_{*} \{1, h_{10}, \xi, h_{10} \xi, b_{11}, h_{11} \xi\}$. Proposition 8.4, Lemmas 8.7 and 8.8 show $d_5(x) = 0$ for $x \in F'$ in $E_*^5 (L_2 V_2)$, since $i_* (xb_{10}^2) = 0$ in (9.2). Therefore we see that
\[ E_6^*(L_2 V_2) = E_{\infty}(L_2 V_2). \] Besides, the extension is trivial, since \( L_2 V_2 \) is an \( M \)-module spectrum.

Now turn to the homotopy groups \( \pi_*(L_2 V_3) \). By Theorem 5.8 and Proposition 6.5, we see that \( E_2^*(L_2 V_3) = E_2^*(L_2 V) \oplus E_2^*(L_2 V) g_3 \). Therefore, by degree reason, we see the following

**Lemma 9.5.** \( E_6^*(L_2 V_3) = E_6^*(L_2 V) \oplus E_6^*(L_2 V) g_3 \).

In the same way as Lemma 6.7, we obtain the following

**Lemma 9.7.** Let \( \otimes \) and note that \( filt \)

\[ \text{Lemma 9.7.} \quad E_6^*(L_2 V_3) = E_6^*(L_2 V) \oplus E_6^*(L_2 V) g_3. \]

In the same way as Lemma 6.7, we obtain the following

**Lemma 9.3.** For \( x \in P \otimes F \otimes \Lambda(\zeta_2) \) with \( d_5(x) = 0 \),

\[ d_6(x g_3) = b_3^3 x \in E_6^*(L_2 V_3). \]

**Lemma 9.5.** Let \( w g_3 \in E_2^*(L_2 V_3) \) denote a non-zero element with \( w \in P \otimes F \otimes \Lambda(\zeta_2) \). If \( d_r(x) = w g_3 \neq 0 \), then \( r = 5 \).

**Proof.** Note that \( d_s(w g_3) = 0 \) for \( s \geq 2 \) by the assumption that \( d_r(x) = w g_3 \). If \( r > 5 \), \( w g_3 \) survives out of \( d_5 \), and so Lemma 9.4 shows that \( d_6(w g_3) = b_3^3 w \) in \( E_6^*(L_2 V_3) \). Therefore \( b_3^3 w \) must be 0 in \( E_6^*(L_2 V_3) \), that is, there exists an element \( y \in E_6^*(L_2 V_3) = E_6^*(L_2 V_3) \) such that \( d_5(y) = b_3^3 w \) in \( E_6^*(L_2 V_3) \). If \( y = y_0 b_3^3 w \) with \( y_0 \in F \otimes \Lambda(\zeta_2) \) and \( u \geq 0 \), then \( d_5(y_0) = b_3^3 v_0 \) for \( v_0 \) with \( v = b_3^3 v_0 \), since \( b_3^3 \) acts monomorphically on \( E_6^*(L_2 V_3) \) by Theorem 5.8 and the naturality of the Adams-Novikov differential. In fact, \( b_3^3 \) detects the homotopy element \( \beta_1 \) by Lemma 2.4.

The assumption \( w g_3 \neq 0 \) in \( E_2^*(L_2 V_3) \) shows that \( u < 3 \), since \( d_5(y_0 b_3^3 g_3) \neq w g_3 \) otherwise, which is verified by sending it under the map \( V_3 \to \Sigma^3 V \) by using the naturality \( d_6(\chi b_3^3) = d_6(\chi b_3^3) \). Recall (2.7) the definition of the Adams-Novikov filtration \( filt \) and note that \( filt y_0 \leq 5 \), since \( y_0 \in F \otimes \Lambda(\zeta_2) \). Then \( filt b_3^3 y_0 \leq 10 \), and so \( filt w g_3 \leq 8 \). On the other hand, if \( r > 5 \), then \( r > 8 \) by degree reason, and \( filt w g_3 > 8 \). This is a contradiction. \[ \text{q.e.d.} \]

**Lemma 9.6.** If \( w \in b_3^3(P \otimes F \otimes \Lambda(\zeta_2)) \subset E_2^*(L_2 V_3) \), then \( w \) does not survive to \( E_6^*(L_2 V_3) \).

**Proof.** Put \( w = b_3^3 x \) for \( x \in P \otimes F \otimes \Lambda(\zeta_2) \). If \( d_5(x) \neq 0 \), then \( d_5(w) \neq 0 \) and \( w \) dies. Otherwise, Lemma 9.4 shows \( d_6(x g_3) = w \) and \( w \) is killed. \[ \text{q.e.d.} \]

**Lemma 9.7.** \( E_7^*(L_2 V_3) \cong E_{\infty}^*(L_2 V_3) \).

**Proof.** Suppose that \( d_r(x) = y \) for \( r \geq 7 \). If \( y \neq 0 \), then \( y = y_0 b_3^3 \) with \( y_0 \in F \otimes \Lambda(\zeta_2) \) and \( u < 3 \) by Lemmas 9.5 and 9.6. By degree reason, \( r \geq 9 \), and \( filt y \leq 4 \). Therefore, the only possibility supporting a non-trivial differential is \( d_9(v_2^2) = v_2 y_0 b_3^3 y_1 \zeta_2 \), which does not happen by degree reason. In fact, \( |d_9(v_2^2)| = 15 \) (9) and \( |v_2^2 b_3^3 y_1 \zeta_2| = 11 \) (9). Therefore, \( d_r(x) = 0 \) for \( r \geq 7 \). \[ \text{q.e.d.} \]

**Remark 9.8.** For an element \( x \in P \otimes F \otimes \Lambda(\zeta_2) \subset E_2^*(L_2 V_3) \), \( x \) is a permanent cycle if \( d_5(x) = 0 \).
Proposition 9.9. In the Adams-Novikov spectral sequence for $L_2V_3$, there exists an element $l \in \mathbb{Z}/3$ such that $d_5(v_2^{j+3l}\zeta_2) = 0$. Furthermore,

$$
d_5(v_2^{j+3l}\zeta_2) = \begin{cases} 
0, & j \equiv 0, 1, 5 \pmod{9}, \\
-\lambda v_2^{j+3l-2}h_{11}b_{10}^2\zeta_2, & j \equiv 3, 4, 8 \pmod{9}, \\
\lambda v_2^{j+3l-2}h_{11}b_{10}^2\zeta_2, & j \equiv 2, 6, 7 \pmod{9}.
\end{cases}
$$

$$
d_5(v_2^{j+3l}h_{11}\zeta_2) = 0,
$$

$$
d_5(v_2^{j+3l}b_{11}\zeta_2) = 0.
$$

$$
d_5(v_2^{j+3l}b_{11}\zeta_2) = \begin{cases} 
\lambda v_2^{j+3l}h_{10}b_{10}^3\zeta_2, & j \equiv 0, 1, 5 \pmod{9}, \\
0, & j \equiv 3, 4, 8 \pmod{9}, \\
-\lambda v_2^{j+3l}h_{10}b_{10}^3\zeta_2, & j \equiv 2, 6, 7 \pmod{9}.
\end{cases}
$$

Proof. Suppose that $d_5(\zeta_2) = xv_2^{-2}h_{11}b_{10}^2\zeta_2$ in $E_2^*(L_2V)$ by Lemma 8.8. Then the coefficients of $d_5(v_2^3\zeta_2)$ and $d_5(v_2^3\zeta_2)$ are $x - \lambda$ and $x + \lambda$, respectively, by Lemma 8.7. Since $\lambda = \pm 1$, one of $x, x - \lambda$ and $x + \lambda$ is zero as desired.

It suffices to show the equation on $d_5(v_2^3\zeta_2)$ by Lemma 8.7. Consider the inclusion $V \to V_3$, and Theorem 2.6 shows that $v_2^3$ is permanent if $j \equiv 0, 1, 5 \pmod{9}$ in the Adams-Novikov spectral sequence for $L_2V_3$ as well. Take $l \in \mathbb{Z}/3$ so that $d_5(v_2^{j+3l}\zeta_2) = 0$ as shown above. Then Lemma 8.7 shows the case for $j \equiv 0, 3, 6 \pmod{9}$. The Bockstein operation $\delta$ acts also trivially on $v_2^{j+3l}\zeta_2$, in fact, $\delta(v_2^{j+3l}\zeta_2) = 0$ and $\delta(\zeta_2) = 0$. Now we can apply Lemma 1.9 to get

$$
d_5(v_2^{j+3l}\zeta_2) = i_{1*}(\omega(v_2^{j+3l}))\delta(v_2^{j+3l}\zeta_2) = 0
$$

for $j \equiv 0, 1, 5 \pmod{9}$, since $\delta(v_2^{j}\zeta_2) = 0$ and $\partial(v_2^{j+3l}\zeta_2) \neq 0$ in these cases. The others follow again from Lemma 8.7 immediately.

q.e.d.

Proposition 9.10. In the Adams-Novikov spectral sequence for $L_2V_3$, there exists an element $k \in \mathbb{Z}/3$ such that $d_5(v_2^{j+3k}\psi_0) = 0$. Using $k$ we can state the differential as follows:

$$
d_5(v_2^{3k+j}\psi_0) = \begin{cases} 
0, & j \equiv 0, 4, 8 \pmod{9}, \\
-\lambda v_2^{3k+j-3}\xi b_{11}b_{10}^2, & j \equiv 2, 3, 7 \pmod{9}, \\
\lambda v_2^{3k+j-3}\xi b_{11}b_{10}^2, & j \equiv 1, 5, 6 \pmod{9},
\end{cases}
$$

$$
d_5(v_2^j\xi) = 0,
$$

$$
d_5(v_2^j\xi) = 0,
$$

$$
d_5(v_2^{3k+j}\psi_1) = \begin{cases} 
0, & j \equiv 3, 4, 8 \pmod{9}, \\
-\lambda v_2^{3k+j-1}\xi b_{10}^3, & j \equiv 2, 6, 7 \pmod{9}, \\
\lambda v_2^{3k+j-1}\xi b_{10}^3, & j \equiv 0, 1, 5 \pmod{9},
\end{cases}
$$

$$
d_5\left(v_2^{3k+3l+j}\psi_0\zeta_2\right) = \begin{cases} 
0, & j \equiv 0, 4, 8 \pmod{9}, \\
-\lambda v_2^{3k+3l+j-3}\xi b_{11}b_{10}^2\zeta_2, & j \equiv 2, 3, 7 \pmod{9}, \\
\lambda v_2^{3k+3l+j-3}\xi b_{11}b_{10}^2\zeta_2, & j \equiv 1, 5, 6 \pmod{9},
\end{cases}
$$

$$
d_5(v_2^j\xi) = 0,
$$

$$
d_5(v_2^j\xi) = 0,
$$

$$
d_5(v_2^{3k+3l+j}\psi_1\zeta_2) = \begin{cases} 
0, & j \equiv 3, 4, 8 \pmod{9}, \\
-\lambda v_2^{3k+3l+j-1}\xi b_{10}^3\zeta_2, & j \equiv 2, 6, 7 \pmod{9}, \\
\lambda v_2^{3k+3l+j-1}\xi b_{10}^3\zeta_2, & j \equiv 0, 1, 5 \pmod{9},
\end{cases}
$$

for $l \in \mathbb{Z}/3$ of Proposition 9.9.
Proof. Let $\delta$ denote the Bockstein operation in (1.8). By degree reason, $\psi_0^3 = v_2^2 \psi_0$ in the cobar complex, and so $0 = \delta(\psi_0^3) = v_2^2 \delta(\psi_0) = v_2^2 \delta(\psi_0) - b_{11} \xi$. Thus

\[(9.11) \quad \delta(\psi_0) = v_2^{-2} b_{11} \xi.\]

This implies

\[(9.12) \quad \delta(v_2^2 \psi_0) = 0\]

by Proposition 5.9. As in the proof of Proposition 9.9, we obtain an element $k$ of $\mathbb{Z}/3$ such that $d_5(v_2^{3k} \psi_0) = 0$ from Lemma 8.6. Now Corollary 8.5 is applied to show

\[d_5(v_2^{3k+1} \psi_0) = \lambda v_2^{3k-2} b_{11} \xi b_{10}^2\]

under (9.11), since $h_{11} \xi = 0$ by Proposition 5.9. Therefore, we have the proposition for $3k + j$ with $j = 0, 3, 6, 1, 4, 7$ (9) by Lemma 8.6.

By Lemma 8.6, one of $d_5(v_2^{3k+2+3m} \psi_0)$ for $m = 0, 1, 2$ is zero. Suppose now that $d_5(v_2^{3k+2+3m} \psi_0) = 0$. Corollary 8.5 and (9.12) show $d_5(v_2^{3k+3+3m} \psi_0) = 0$, which contradicts the previous result if $m = 0, 1$. Therefore, $d_5(v_2^{3k+8} \psi_0) = 0$. With Lemma 8.6, we obtain the first four equations.

As we have seen above, $v_2^{3k+1} \psi_0$ is permanent by Remark 9.8 if $j = 0, 4, 8$ (9). $v_2^{3k} \zeta_2$ is also permanent by Proposition 9.9. We then apply Lemma 1.9 to show

\[d_5(v_2^{3k+3l} \psi_0 \zeta_2) = \iota_{14}(\omega(v_2^{3k+1} \psi_0)) \delta(v_2^{3k} \zeta_2) = 0.\]

In fact, $\delta(v_2^{3k+3l} \zeta_2) = 0$ and $\delta(v_2^{3k+3l} \psi_0 \zeta_2) = v_2^{3k+3l} \zeta_2 \delta(v_2^{3k} \psi_0) \neq 0$ for $j = 0, 4$ (9) by (9.11) and Proposition 5.9.

By Lemma 8.7, one of $d_5(v_2^{3k+3l+3m+2} \psi_0 \zeta_2)$ is zero for $m = 0, 1, 2$. Then again apply Corollary 8.5 and (9.12) to see $d_5(v_2^{3k+3l+3m+3} \psi_0 \zeta_2) = 0$, which contradicts the previous case if $m = 0, 1$, and so $m = 2$.

Lemmas 8.7 and 8.8 show the other equations. q.e.d.

By Lemma 9.3, we obtain

**Proposition 9.13.**

1) The equations in Propositions 9.9 and 9.10 hold true in $E_7^*(L_2 V)$.

2) For an equation $d_5(x) = y$ in Propositions 8.4, 9.9 and 9.10, $d_5(x g_3) = y g_3$ in $E_7^*(L_2 V_3)$.

To describe the $E_7$-term, we introduce the notations:

- $K = \mathbb{Z}/3[v_0^3, v_2^{-3}]$,
- $V = \mathbb{Z}/3[1, v_2, v_2^3] / \mathcal{V} = \mathbb{Z}/3[v_2^3, v_2, v_0^3, v_2, v_0^7, v_0^5]$,
- $P = \mathbb{Z}/3[b_0], P_k = P/(b_k^k) = \mathbb{Z}/3[b_0]/(b_k^k)$,
- $F_1 = \mathbb{Z}/3[1, v_2^3 h_{10}, v_2^{-2} h_{11}, v_2^3 b_{11}, v_2^{3k-1} \psi_0, v_2^{3k+3} \psi_0, v_2^{3k+1} \xi, v_2^{3k-4} \xi b_{11}]$,
- $F_2 = \mathbb{Z}/3[v_2^3 h_{10}, v_2^{3k-4} \xi b_{11}]$ and $F_3 = \mathbb{Z}/3[v_2^3 h_{10}, v_2^{3k+2} \xi]$, for $k$ and $l$ in $\mathbb{Z}/3$ given in Propositions 9.9 and 9.10.

Then Propositions 8.4, 9.9, 9.10 and 9.13 imply the following

**Proposition 9.14.** The $E_7$-term of the Adams-Novikov spectral sequence for $L_2 V_3$ are isomorphic to the tensor product of the $K$-module $\Lambda(v_2^{3l} \zeta_2, g_3)$ and the $K$-module of the direct sum of

- $K \otimes V \otimes P \otimes F_1$,
- $K \otimes \mathcal{V} \otimes P_2 \otimes F_2$ and
- $K \otimes \mathcal{V} \otimes P_3 \otimes F_3$.

Here $l \in \mathbb{Z}/3$ is the element given in Proposition 9.9.
Now Lemmas 9.4 and 9.7 show

**Theorem 9.15.** The homotopy groups of $L_2V_3$ is isomorphic to the tensor product of $K$-module $\Lambda(v_2^3)\zeta_2$ and the $K$-module of the direct sum of

$$K \otimes V \otimes P_3 \otimes F_1, \quad K \otimes V \otimes P_2 \otimes F_2 \otimes \Lambda(g_3) \quad \text{and} \quad K \otimes V \otimes P_3 \otimes F_3 \otimes \Lambda(g_3).$$

**Proof.** The module structure of $E_7(L_2V_3)$ is read off from Proposition 9.14 and Lemma 9.4, which gives the $E_\infty$-term by Lemma 9.7. Since $V_3$ is an $M$-module spectrum, the extensions are trivial.

q.e.d.

Propositions 8.4 and 9.13 also show

**Proposition 9.16.** The $E_6$-term of the Adams-Novikov spectral sequence for $L_2V$ is isomorphic to the tensor product of the $K$-module $\Lambda(v_2^3)\zeta_2$ and the $K$-module of the direct sum of

$$K \otimes V \otimes P \otimes F_1, \quad K \otimes V \otimes P_2 \otimes F_2 \quad \text{and} \quad K \otimes V \otimes P_3 \otimes F_3.$$

10. The $E_{10}$-term of the Adams-Novikov spectral sequence for $L_2V$

In this section we compute $d_9 : E_9^0(L_2V) \to E_{9+9}^0(L_2V)$, in fact, $E_9^0(L_2V) = E_9^0(L_2V)$ by degree reason.

**Lemma 10.1.** In the Adams-Novikov spectral sequence for computing $\pi_*(L_2V)$,

$$d_9(v_2^j h_{10}) = v_2^{j-3} b_{10}^5, \quad j = 3, 4, 8 (9), \quad \text{and} \quad d_9(v_2^{3j} h_{10} \zeta_2) = v_2^{3j-3} b_{10}^5 \zeta_2, \quad j = 3, 4, 8 (9),$$

up to sign, for $l$ of Proposition 9.9.

**Proof.** In this proof, the equations are all up to sign. Since $\beta_1^6 = 0$ in $\pi_*(S^0)$ (cf. [20]),

$$xb_{10}^6 \text{ is killed in } E_9^*(L_2V) \text{ for any permanent cycle } x \in E_9^2(L_2V).$$

As we have seen that $v_2^j$ is a permanent cycle for $j = 0, 1, 5 (9)$ in Theorem 2.6, $v_2^j b_{10}^6$ must be killed by virtue of (10.2). Comparing the degrees in sight of Theorem 5.8, either

$$d_9(v_2^{j+3} \psi_0) = v_2^j b_{10}^6 \quad \text{or} \quad d_9(v_2^{j+3} h_{10} b_{10}) = v_2^j b_{10}^6$$

holds. If $j \equiv 0 (9)$ for $k = 0$, if $j \equiv 1$ or $5 (9)$ for $k = 1$, or if $k = 2$, then $v_2^{j+3} \psi_0$ dies in $E_9^0(L_2V)$ by Proposition 9.10, where $k$ is an element of $\mathbb{Z}/3$ given in Proposition 9.10. Therefore, the only choice is $d_9(v_2^{j+3} h_{10} b_{10}) = v_2^j b_{10}^6$ in this case.

Suppose the other case where $j \equiv 1$ or $5 (9)$ if $k = 0$, and $j \equiv 0 (9)$ if $k = 1$. Suppose further that $d_9(v_2^{j+3} \psi_0) = v_2^j b_{10}^6$, and multiply it by $h_{10}$ to get

$$d_9(v_2^{j+3} \psi_0 h_{10}) = v_2^j h_{10} b_{10}^6,$$

which equals

$$d_9(v_2^{j+3} \zeta_2 b_{10}) = v_2^j h_{10} b_{10}^6$$

by the relation $\psi_0 h_{10} = -b_{10}^{10}$ of Proposition 5.9. Since $d_9(v_2^j b_{11}) = \pm v_2^j h_{10} b_{10}^3$ by Proposition 8.4, we see that $d_9(v_2^{j+3} \zeta_2 b_{10}) = v_2^j h_{10} b_{10}^6 = d_9(v_2^j b_{11} b_{10}^3) = 0$ in $E_9^0(L_2V)$. Therefore, $d_9(v_2^{j+3} \zeta_2) = 0$ by Lemma 7.1. If $d_{13}(v_2^{j+3} \zeta_2) \neq 0$, then $d_{13}(v_2^{j+3} \zeta_2) = v_2^{j-1} b_{10}^7 \zeta_2$ by Theorem 5.8, since it is the only choice. Suppose that
$v_2^j b_{10}^j \neq 0$ in $E_{13}^{12}(L_2 V)$. This implies that $v_2^{j-1} \zeta_2$ is a permanent cycle by Lemma 7.1, and so $v_2^{j-1} b_{10}^{j-1} \zeta_2$ must be killed by virtue of (10.2). This is a contradiction. In fact, if $d_r(x) = v_2^{j-1} b_{10}^{j-1} \zeta_2$, then $d_r(x b_{10}) = v_2^{j-1} b_{10}^{j-1} \zeta_2$. By the supposition, $r \geq 13$, and there is no candidate for $x$ by Theorem 5.8. Therefore, $d_{13}(v_2^{j+3} \xi) = 0$. Again use (10.2) to see that $v_2^{j+3} \xi b_{10}^j$ must be killed in the spectral sequence for $E_{19}^{7} V$. Theorem 5.8 and Lemma 6.9 show that the only possibility is

$$d_{13}(v_2^{j+7} h_{10}) = v_2^{j+3} b_{10}^6 \xi.$$  

Theorem 5.8 also shows that $v_2^{j+3} b_{10}^9 \xi$ is not killed by $d_9$. Therefore the above equation implies

$$d_{13}(v_2^{j+7} h_{10} b_{10}^3) = v_2^{j+3} b_{10}^9 \xi \quad \text{in} \quad E_{13}^{20}(L_2 V),$$

which contradicts to Proposition 9.16 in which $v_2^{j+7} h_{10} b_{10}^3$ is shown not to be $E_6$-term for $j = 0, 1, 5$ (9). Therefore, the only choice is also $d_9(v_2^{j+3} h_{10} b_{10}) = v_2^{j+3} b_{10}^9$ in this case, which shows the first equation.

By the same argument as above, we obtain the other ones. q.e.d.

**Corollary 10.3.** For the differential $d_9$,

$$d_9(v_2^j h_{11}) = v_2^{j-3} b_{10}^1 b_{10}^j, \quad j = 3, 7, 8 (9), \quad \text{and}$$

$$d_9(v_2^{j+3} h_{11} \zeta_2) = v_2^{j+3} b_{10}^1 b_{10}^j \zeta_2, \quad j = 3, 7, 8 (9).$$

**Proof.** Since $v_2^2 b_{11}$ is a permanent cycle and originates from the sphere, the first equation of Lemma 10.1 with $v_2^3 b_{11}$ gives rise to $d_9(v_2^{j+3} h_{10} b_{11}) = v_2^{j+3} b_{10}^1 b_{10}^j$, and the first one since $h_{10} b_{11} = -v_2 h_{11} b_{10}$ in Proposition 5.9. By the same reason, we obtain the second half. q.e.d.

**Corollary 10.4.** The element $l$ of $Z/3$ in Proposition 9.9 is zero. That is, $d_5(v_2^j \zeta_2) = 0$ for $j = 0, 1, 5 (9)$.

**Proof.** Since $v_2 h_{11} \zeta_2$ is a permanent cycle by Lemma 2.4, Corollary 10.3 indicates that $j \neq 3, 7, 8 (9)$ if $j + 3l = 1 (9)$. Therefore, $l \neq 1$. Similarly, $v_2^j h_{11} \zeta_2$ is permanent, and so $j \neq 3, 7, 8 (9)$ if $j + 3l = 4 (9)$. This shows that $l \neq 2$. q.e.d.

**Proposition 10.5.** In the Adams-Novikov spectral sequence for $\pi_*(L_2 V)$,

$$d_9(v_2^j h_{10}) = v_2^{j-3} b_{10}^1, \quad j = 3, 4, 8 (9),$$

$$d_9(v_2^j h_{11}) = v_2^{j-4} b_{10}^1 b_{10}^1, \quad j = 3, 7, 8 (9),$$

$$d_9(v_2^{j+3} \xi) = v_2^{j+3} \psi_0 b_{10}^4, \quad j = 2, 3, 7 (9), \quad \text{and}$$

$$d_9(v_2^{j+3} \zeta_2) = v_2^{j+3} \zeta_2 b_{10}^1, \quad j = 1, 5, 6 (9); \quad \text{and}$$

$$d_9(v_2^j h_{10} \zeta_2) = v_2^{j-3} b_{10}^1 \zeta_2, \quad j = 3, 4, 8 (9),$$

$$d_9(v_2^j h_{11} \zeta_2) = v_2^{j-4} b_{10}^1 b_{10}^1 \zeta_2, \quad j = 3, 7, 8 (9),$$

$$d_9(v_2^{j+3} \xi \zeta_2) = v_2^{j+3} \psi_0 b_{10}^4 \zeta_2, \quad j = 2, 3, 7 (9), \quad \text{and}$$

$$d_9(v_2^{j+3} \zeta_2 \zeta_2) = v_2^{j+3} \zeta_2 b_{10}^1 \zeta_2, \quad j = 1, 5, 6 (9)$$

up to sign for $k \in Z/3$ in Proposition 9.10.

**Proof.** We have seen the first, the second, the fifth and the sixth equations by Lemma 10.1 and Corollary 10.3.
By Theorem 5.8, for $j = 3k + m$ with $m \equiv 0, 4, 8$ (9),

\[
d_9(v^j_1v^0_0) \in \mathbb{Z}/3\{(v^j_2b^6_{10}, v^3_2h^5_{10}b^2_{10}), v^{j-3}_2(\psi_1b^4_{10}c_2)\text{ and } d_9(v^j_2v^0_0) \in \mathbb{Z}/3\{(v^j_2b^6_{10}c_2).}
\]

By Proposition 9.16 and the first and the fifth equations, we see that $v^j_2b^6_{10}$ and $v^j_2b^6_{10}c_2$ are killed by $d_5$ or $d_9$ for any $j$. The element $v^{j-3}_2h^5_{10}b^2_{10}$ is similarly seen to be away from the $E_{10}$-term. The other element $v^j_2(\psi_1b^4_{10}c_2)$ is also zero in the $E_6$-term by Proposition 9.16 in this case. Therefore, $d_9(v^j_2v^0_0) = 0$ and $d_9(v^j_2v^0_0) = 0$. Hence, $v^j_2v^0_0$ and $v^j_2v^0_0c_2$ are permanent cycles of the Adams-Novikov spectral sequences for $\pi_*(V \wedge \mathbb{E}_{16})$ and $\pi_*(V \wedge \mathbb{E}_{17})$, respectively, and $v^j_2v^0_0b^6_{10}$ and $v^j_2v^0_0b^6_{10}c_2$ must be killed by virtue of (10.2). By Lemma 6.9, $v^j_2h^5_{10}c_2$ cannot be a killer of $v^j_2v^0_0b^6_{10}$. Proposition 9.16 says that $v^{j+1}_2b^6_{10}$ and $v^{j+1}_2b^6_{10}c_2$ are dead in the $E_6$-term for $j = 3k + m$ with $m \equiv 0, 4, 8$ (9). Therefore, we have the only choice $d_9(v^{j+3}_2\xi b^4_{10}) = v^j_2v^0_0b^6_{10}$ and $d_9(v^{j+3}_2\xi b^6_{10}c_2) = v^j_2v^0_0b^6_{10}c_2$, and so

\[
d_9(v^{j+3}_2\xi) = v^j_2v^0_0b^4_{10} \text{ and } d_9(v^{j+3}_2\xi b^6_{10}c_2) = v^j_2v^0_0b^4_{10}c_2
\]

for $j \equiv 3k + m$ with $m \equiv 0, 4, 8$ (9) by Proposition 9.16. Now $\beta_{6/4} = b^3_{11}$ sends these to $d_9(v^{j+6}_2b^6_{11}) = v^{j+3}_2b^6_{11}v^0_0b^4_{10} = v^{j+4}_2(\psi_1b^4_{10}c_2 + \xi b^6_{10})$ and $d_9(v^{j+6}_2b^6_{11}c_2) = v^{j+4}_2(\psi_1b^4_{10}c_2 + \xi b^6_{10}c_2)$ as desired.

These imply the following

**Theorem 10.6.** The $E_{10}$-term of the Adams-Novikov spectral sequence for $\pi_*(L_2V(1))$ is isomorphic to the tensor product of $\Lambda(\xi_2)$ and the direct sum of $P$-modules

\[
\begin{align*}
K \otimes V \otimes P_1 & \otimes \{v^3_2b^1_{11}, v^{3k-1}_2v^0_0\}, \\
K \otimes V \otimes P_5 & \otimes \{1, v^{3k+3}_2\psi_1\}, \\
K \otimes \mathbb{V} \otimes P_2 & \otimes \{v^2_2b^1_{11}, v^{3k-4}_2\xi b^1_{11}\} \quad \text{and} \\
K \otimes \mathbb{V} \otimes P_3 & \otimes \{v^3_2b^1_{10}, v^{3k+2}_2\xi\}.
\end{align*}
\]

**Corollary 10.7.** $\pi_*(L_2V) \cong E_{10}(L_2V)$.

**Proof.** Theorem 10.6 shows $E^s_{10}(L_2V) = 0$ if $s > 12$, and we see that $d_{14} = 0$. Therefore, $E_{12}(L_2V) = E_{10}(L_2V)$. Since $L_2V$ is an $M$-module spectrum (cf. [27]), the homotopy groups of $L_2V$ are a $\mathbb{Z}/3$-vector space, and so there arises no extension problem, and we have the corollary.

q.e.d.

11. The Non-Existence of $\beta$-Elements

The definition (2.1) of $\beta$-elements gives us a representative of each $\beta$-element in the cobar complex, which we have in [14, Lemma 4.4]:

\[
\beta_{3k+1} = -v^3_2b^{10} + \cdots \text{ and } \beta_{3k+2} = -v^3_2(t^2 \otimes t^3 - t_1 \otimes t^3_c + \cdots).\]

Consider the composition

\[
f : S^0 \xrightarrow{\eta} L_2S^0 \xrightarrow{L_2i_i} L_2V(1),
\]

where $\eta : S^0 \to L_2S^0$ is the localization map, and $i$ and $i_1$ are maps of (1.5) and (1.6).

**Proposition 11.1.** In the Adams-Novikov spectral sequence for computing $\pi_*(S^0)$, $\beta_t \in E^2_2(S^0)$ dies under $d_5$ if $t = 4, 7$ (9), and under $d_9$ if $t = 8$ (9).
Proof. By (2.1), \( f_*(\beta_{3s+1}) = v_2^{3s}b_{10} \) and \( f_*(\beta_{3s+8}) = v_2^{9s+7}h_{11}\zeta_2 \). Then for the case where \( t = 4, 7 \) (9), we compute \( f_*(d_5(\beta_{3s+1})) = d_5(v_2^{3s})b_{10} \) by the naturality, which is not zero by Proposition 8.4 unless \( s \equiv 0 \) (3). Therefore, \( d_5(\beta_{3s+1}) \neq 0 \) in this case. In the same way, \( f_*(d_5(\beta_{3s+8})) = d_5(v_2^{9s+7}h_{11}\zeta_2) \neq 0 \) by Corollaries 10.3 and 10.4.

q.e.d.

We next consider more \( \beta \)-elements. Recall [12] the definition of \( \beta \)-elements in the \( E_2 \)-term of the Adams-Novikov spectral sequence for \( S^0 \). In \( v_2^{-1}BP_* \), Miller, Ravenel and Wilson introduce elements \( x_i \) such that

\[
d(x_i) = \pm v_1^{3i-1}v_2^2 t_1 \mod (3, v_1^{4i-1})
\]

for the differential \( d : v_2^{-1}BP_* \to v_2^{-1}BP_*(BP) \) defined by \( d = \eta_R - \eta_L \). These are restated as elements of \( E(2)_s \):

\[
x_i = v_2^{2i} \quad \text{(for } i \leq 1), \quad x_2 = v_2^3 - v_2^5 v_2^7 \quad \text{and}
\]

\[
x_i = x_{i-1}^{3} + v_1^{4(3i-1)} v_2^{23i-1+1} \quad \text{(for } i \geq 3).
\]

By (11.2), we see that

\[
x_i^s \in \text{Ext}^0(A/(3, v_1^i))
\]

for \( j < 4 \cdot 3^{i-1} \). Now consider the connecting homomorphisms

\[
\partial_j : \text{Ext}^0(A/(3, v_1^i)) \to \text{Ext}^1(A/(3)) \quad \text{and} \quad \partial : \text{Ext}^1(A/(3)) \to \text{Ext}^2(A)
\]

associated to the short exact sequences \( 0 \to A/(3) \to A/(3) \to A/(3, v_1^i) \to 0 \) and \( 0 \to A \to A \to A/(3) \to 0 \). Then \( \beta \)-elements are defined by (cf. [12])

\[
\beta_{3s+j} = \partial \partial_j(x_i^s) \in \text{Ext}^2(A)
\]

for \( j < 4 \cdot 3^{i-1} \) and \( s > 1 \), and at \( s = 1 \),

\[
\beta_{3s/j} = \partial \partial_j(v_2^{3i}) \in \text{Ext}^2(A)
\]

for \( j \leq 3^i \). As usual, \( \beta_s = \beta_s/1 \). This gives us an idea to define another \( \beta \)-element:

\[
\beta_{3s/3^i} = \partial \partial_j(3^i) \in \text{Ext}^2(A)
\]

for \( j \leq 3^i \) and \( s > 0 \). In order to distinguish these \( \beta \)-elements, we denote the \( \beta \)-elements of (11.4) by \( \beta_{3s/3^i}^j \) for \( j \leq 3^i \) and \( s > 0 \). Then the \( \beta \)-elements of (11.5) give rise to

\[
f_*(\beta_{3s/3^i}) = \begin{cases} [sv_1^{3b_{i-s-3^i+6}b_{10}}], & \text{if } i \text{ odd } > 1, \\ [sv_1^{3b_{i-s-2^i-2}b_{11}}], & \text{if } i \text{ even } > 0. \end{cases}
\]

Here \( b_{i,s} = (3^i - 1)/4 + 3^i(s - 1) \). Therefore, Proposition 8.4 shows

**Theorem 11.6.** \( \beta_{3t+3/3^3} \) and \( \beta_{3s/3^3} \) die under \( d_5 \) in \( E_2^*(S^0) \) for \( t \geq 0, i > 1 \) and \( s \neq 0 \) (3).

In particular, we have Ravenel’s odd primary Kervaire invariant theorem [18]:

**Corollary 11.7.** In the Adams-Novikov spectral sequence at \( p = 3 \), \( d_5(\beta_{3^i/3^3}) \neq 0 \) for \( i > 0 \).

An additional result is obtained from [18]

**Corollary 11.8.** For the \( \beta \)-element \( \beta_{3t+3/3^3} \in E_2^*(S^0) \),

\[
d_5(\beta_{3t+3/3^3}) \neq 0
\]

for \( t \neq 0 \) (9).
By (11.3), (11.4) and (11.5), we have a relation between these $\beta$-elements:

$$f_*(\beta^L_{j,s/j}) = f_*(\beta^L_{j',s/j'}) - f_*(\beta^L_{(9s-2)^{-2}/j-3^i-3^j})$$

for $j \leq 3^i$. Furthermore, (11.2) shows

$$f_*(\beta^L_{j',s/j}) = 0$$

for $i \geq 2$, $s > 0$ and $j \leq 3^i$. Therefore, we cannot tell from our results whether the three primary Kervaire invariant elements $b_1$ of the Adams spectral sequence is permanent. In fact, for example, $b_{12}$ is obtained from $\beta^L_{9/9} \pm \beta^L_{7} \in \pi_*(S^8)$, which is $\beta^L_{9/9}$ and $f_*(\beta^L_{9/9}) = 0 \in \pi_*(L_2V(1))$.

References


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