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Geometry and Quantum Field Theory

EDWARD WITTEN

1. Introduction. First of all, I would like to thank the American Mathematical Society for inviting me to lecture here on this occasion, and to thank the organizers for arranging such a stimulating meeting. And I would like to echo the sentiments of some previous speakers, who expressed the wish that we will all meet here in good health on the 150th anniversary of the American Mathematical Society, to hear the younger mathematicians explain the solutions of some of the unsolved problems posed this week.

It is a challenge to try to speak about the relation of quantum field theory to geometry in just one hour, because there are certainly many things that one might wish to say. The relationship between theoretical physics and geometry is in many ways very different today than it was just ten or fifteen years ago. It used to be that when one thought of geometry in physics, one thought chiefly of classical physics—and in particular of general relativity—rather than quantum physics. Geometrical ideas seemed (except perhaps to some visionaries) to be far removed from quantum physics—that is, from the bulk of contemporary physics. Of course, quantum physics had from the beginning a marked influence in many areas of mathematics—functional analysis and representation theory, just to mention two. But it would probably be fair to say that twenty years ago the day to day preoccupations of most practicing theoretical elementary particle physicists were far removed from considerations of geometry.

Several important influences have brought about a change in this situation. One of the principal influences was the recognition—clearly established by the middle 1970s—of the central role of nonabelian gauge theory in elementary particle physics. The other main influence came from the emerging study of supersymmetry and string theory. Of course, these different influences are inter-related, since nonabelian gauge theories have elegant supersymmetric

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generalizations, and in string theory these appear in a fascinating new light. Bit by bit, the study of nonabelian gauge theories, supersymmetry, and string theory have brought new questions to the fore, and encouraged new ways of thinking.

An important early development in this process came in the period 1976–77 with the recognition that the Atiyah-Singer index theorem was the proper context for understanding some then current developments in the theory of strong interactions. (In particular, the solution by Gerard't Hooft [**1**] of the "U(1) problem," a notorious paradox in strong interaction theory, involved Yang-Mills instantons, originally introduced in [**2**], and "fermion zero modes" whose proper elucidation involves the index theorem.) Influenced by this and related developments, physicists gradually learned to think about quantum field theory in more geometrical terms. As a bonus, ideas coming at least in part from physics shed new light on some mathematical problems. In the first stage of this process, the purely mathematical problems that arose (at least, those that had motivations independent of quantum field theory, and in which progress could be made) involved "classical" mathematical concepts—partial differential equations, index theory, etc.—where physical considerations suggested new questions or a new point of view.

In the talk just before mine, Karen Uhlenbeck described some purely mathematical developments that at least roughly might be classified in this area. She described advances in geometry that have been achieved through the study of systems of nonlinear partial differential equations. Among other things, she sketched some aspects of Simon Donaldson's work on the geometry of four-manifolds [**3**], in which dramatic advances have been made by studying the moduli spaces of instantons—solutions, that is, of a certain nonlinear system of partial differential equations, the self-dual Yang-Mills equations, which were originally introduced by physicists in the context of quantum field theory [**2**].

If "classical" objects (such as instantons) that arise in quantum field theory could be so interesting mathematically, one might well suspect that mathematicians will soon find the quantum field theories themselves, and not only the "classical" objects that they give rise to, to be of interest. Such a question was indeed raised by Karen Uhlenbeck at the end of her talk, and is much in line with the perspective offered by Michael Atiyah in [**4**], which was the starting point for many of my own efforts.

I will talk today about three areas of recent interest where quantum field theory seems to be the right framework for thinking about a problem in geometry:

(1) Our first problem will be to explain the unexpected occurrence of modular forms in the theory of affine Lie algebras. This problem, which was described the other day by Victor Kac, has two close cousins—to explain "monstrous moonshine" in the theory of the Fischer-Griess monster group [**5, 6**], and to account for the surprising role of modular forms in algebraic

topology [7], about which Raoul Bott spoke briefly at the end of his talk. Quantum field theory supplies a more or less common explanation for these three phenomena, but the first requires the least preliminary explanation, and it is the one that I will focus on.

(2) The second problem is to give a geometrical definition of the new knot polynomials—the Jones polynomial and its generalizations—that have been discovered in recent years. The essential properties of the Jones polynomial have been described to us the other day by Vaughn Jones.

(3) The third problem is to get a more general insight into Donaldson theory of four-manifolds—which was sketched in the last hour by Karen Uhlenbeck—and the closely related Floer groups of three-manifolds. Here again there are lower dimensional cousins, namely the Casson invariant of three-manifolds, Gromov's theory of maps of a Riemann surface to a symplectic manifold, and Floer's closely related work on fixed points of symplectic diffeomorphisms. But among these formally rather analogous subjects, I will concentrate on Donaldson/Floer theory.

2. Physical Hilbert spaces and transition amplitudes. Let us sketch these three problems in a little more detail. In the first problem, one considers the group $\mathscr{L}G$ of maps $\mathbf{S}^1 \to G$, where G is a finite-dimensional compact Lie group, and $\mathbf{S}^1$ is the ordinary circle. The representations of $\mathscr{L}G$ with "good" properties, analogous to the representations of compact finite-dimensional groups, are the so-called integrable highest weight representations (see [**8**, **9**] for introductions.) These representations are rigid (no infinitesimal deformations). From this it follows that any connected group of outer automorphisms of $\mathscr{L}G$ must act at least projectively on any integrable highest weight representation $\mathscr{R}$ of $\mathscr{L}G$. In fact, the group $\operatorname{diff}\mathbf{S}^1$ of diffeomorphisms of $\mathbf{S}^1$ acts on $\mathscr{L}G$ by outer automorphisms and acts projectively on the integrable highest weight representations. Thus, in particular, the vector field $d/d\theta$ that generates an ordinary rotation of $\mathbf{S}^1$ is represented on $\mathscr{R}$ by some operator H.

One computes in such a representation the "character"

$$F_{\mathscr{R}}(q) = \operatorname{Tr}_{\mathscr{R}} q^H \tag{2.1}$$

(here q is a complex number with $|q| < 1$), and one finds this to be a modular function with a simple transformation law under a suitable congruence subgroup of the modular group. Setting $q = \exp(2\pi i\tau)$, the modular group is of course the group $\mathrm{PSL}(2, \mathbb{Z})$ of fractional linear transformations

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{2.2}$$

of the upper half-plane, with $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. I will not enter here into the complicated question of exactly what kind of modular functions the characters (2.1) are. (One simple, general statement, which from one point of view is the statement that comes most directly from quantum field theory, is that the $F_{\mathscr{R}}(q)$, with $\mathscr{R}$ running over all highest weight representations of fixed "level", transform as a unitary representation of the full modular group $\mathrm{PSL}(2, \mathbb{Z})$.)

To understand the significance of the modularity of the characters $F_{\mathscr{R}}(q)$, let us recall that the group $\mathrm{SL}(2, \mathbb{Z})$ has a natural interpretation as the (orientation preserving) mapping class group of a two-dimensional torus $\mathbf{T}^2$. Thus, we interpret $\mathbf{T}^2$ as the quotient of the $x-y$ plane by the equivalence relations $(x, y) \sim (x+1, y)$ and $(x, y) \sim (x, y+1)$. Clearly, if a, b, c, and d are integers such that $ad-bc=1$, the formula $(x, y) \to (ax+by, cx+dy)$ gives a diffeomorphism of $\mathbf{T}^2$ to itself, and every orientation preserving diffeomorphism of $\mathbf{T}^2$ is isotopic to a unique one of these. Thus, $\mathrm{SL}(2, \mathbb{Z})$ can be considered in this sense to arise as a group of diffeomorphisms of a two-dimensional surface.

Thus, while it is natural that the one-dimensional symmetry group $\operatorname{diff}\mathbf{S}^1$ plays a role in the representation theory of the loop group $\mathscr{L}G$, the appearance of $\mathrm{SL}(2, \mathbf{Z})$ means that in fact a kind of *two-dimensional symmetry* appears in this theory. Our first problem—modular moonshine in the theory of affine Lie algebras—is the problem of explaining the origin of this two-dimensional symmetry.

Now we move on to our second problem. A *braid* is a time dependent history of n points $\mathbb{R}^2$, which are required, up to a permutation, to end where they begin (Figure 1(a)). Braids with n strands form a group, the Artin braid group $\mathscr{B}_n$, with an evident law of composition, sketched in Figure 1(b). From a braid one can make a knot (or in general a link) by gluing together the top and bottom as in Figure 1(c). Although every braid gives in this way a unique link, the converse is not so; the same link may arise from many different braids. The crucial difference between braids and links is the following. Braids are classified up to time dependent diffeomorphisms of $\mathbb{R}^2$ (that is, up to diffeomorphisms of $\mathbb{R}^3$ that leave fixed one of the coordinates, the "time" t), while links are classified up to full three-dimensional diffeomorphisms.

If one is given a representation $\mathscr{S}$ of the braid group $\mathscr{B}_n$, and a braid $B \in \mathscr{B}_n$, then $\mathrm{Tr}_{\mathscr{S}} B$ (the trace of the matrix that represents B in the representation $\mathscr{S}$) is an invariant of the *braid* B (and depends in fact only on its conjugacy class in $\mathscr{B}_n$), but there is no reason for it to be an invariant of the *link* that is obtained by joining the ends of the braid B according to the recipe in the figure.

Nevertheless, Vaughn Jones found a special class of representations of the braid group with the magic property that suitable linear combinations of the braid traces are in fact knot invariants and not just braid invariants. These knot invariants can be combined into the Jones polynomial, some of whose remarkable properties were described in Jones's lecture the other

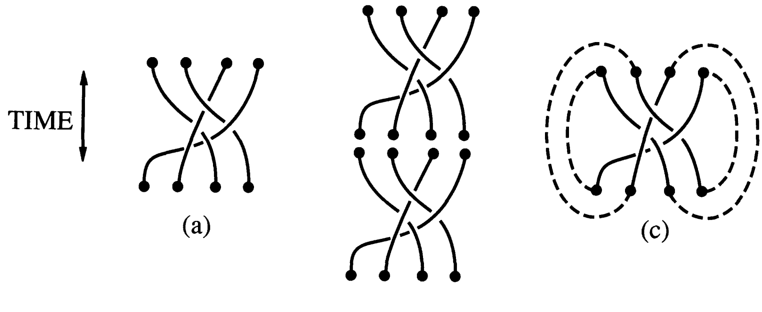

FIGURE 1. (a) A braid; (b) composition of two braids; (c) making a braid into a link.

day. The discovery of the Jones polynomial stimulated in short order the discovery of some related knot polynomials—the HOMFLY and Kauffman polynomials—whose logical status is rather similar. The challenge of understanding the Jones polynomial is to explain why the Jones braid representations, which obviously have two-dimensional symmetry, should really have three-dimensional symmetry.

Thus, we have two examples where one studies a *group representation* that obviously has d-dimensional symmetry, for some d, but turns out to have $(d+1) = D$-dimensional symmetry, for reasons that might look mysterious. In our first example, the group is $\mathscr{L}G$, $d=1$, and $D=2$. In the second example, the group is the braid group, $d=2$, and $D=3$.

Our third example, Donaldson/Floer theory, is of a somewhat different nature. In this case, $d=3$ and $D=4$, but unlike our previous examples, Donaldson/Floer theory began historically not in the lower dimension but in the upper dimension. The mathematical theory begins in this case with Donaldson's invariants of a closed, oriented four-manifold M. In Donaldson's original considerations, it was important that the boundary of M should vanish. The attempt to generalize the Donaldson invariants to the case that $\partial M = Y \neq \varnothing$ led to the introduction of the "Floer homology groups' which are vector spaces $\mathrm{HF}^*(Y)$ canonically associated with an oriented three-manifold Y. Though these vector spaces did not originate as group representations, their formal role is just like that of the group representations that entered in our first two examples.

In our first two problems of understanding modular moonshine and the Jones polynomial, the crucial question is to explain why $(d+1)$-dimensional symmetry is present in a construction that appears to only have d-dimensional symmetry. At least from a historical point of view, Donaldson theory is of a completely different nature, since the four-dimensional symmetry has been built in from the beginning. Nevertheless, the logical structure of Donaldson/Floer theory is of a similar nature to that of the first two examples.

In each of our three examples, a pair of dimensions, d and $D = d+1$, plays a key role. With the lower dimension we associate a vector space (the representations of $\mathscr{L}G$ or $\mathscr{B}_n$ or the Floer groups) and with the upper dimension we associate an invariant (the characters $\mathrm{Tr}_{\mathscr{R}}\, q^H$, the knot polynomials, or the Donaldson invariants of four-manifolds). The facts are summarized in Table 1.

TABLE 1

Theory	Dimensions	Vector space in lower dimension	Invariant in upper dimensions
Modular moonshine	1, 2	Representations of loop groups	Modular forms $\mathrm{Tr}\, q^H$
Jones polynomial	2, 3	Jones representations of braid group	Invariants of knots and three-manifolds
Donaldson/Floer theory	3, 4	Floer Homology groups	Donaldson invariants of four-manifolds

3. Axioms of quantum field theory. Let us now formalize the precise relationship between the vector spaces that appear in dimension d and the invariants in dimension $d+1$. (In the physical context, d is called the dimension of space, and $d+1$ is the dimension of space-time.) In formalizing this relationship, we will follow axioms originally proposed (in the context of conformal field theory, essentially our first example) by Graeme Segal **[10]**. (In addition, Michael Atiyah has adapted those axioms for the topological context that is relevant to our second and third examples **[11]**, with considerably more precision that I will attempt here.)

So we will consider quantum field theory in space-time dimension $D = d+1$. The manifolds that we consider will be smooth manifolds possibly endowed with some additional structure. The type of additional structure considered will be characteristic of the theory. For instance, in the case of modular moonshine, this additional structure is a conformal structure; quantum field theories requiring such a structure (but not requiring a choice of Riemannian metric) are called conformal field theories. In the case of Donaldson/Floer theory, the extra structure consists of an orientation; in the case of the Jones polynomial, one requires an orientation and "framing" of tangent bundles (in a suitable stable sense). Theories that require structure of such a purely topological kind may be called topological quantum field theories. The "ordinary" quantum field theories most extensively studied by physicists require metrics on all manifolds considered.

The first notion is that to every d-dimensional manifold X, without boundary, and perhaps with some additional structure characteristic of the particular theory, one associates a vector space $\mathscr{H}_X$. A quantum field theory is said to be "unitary" if these vector spaces actually carry a Hilbert space structure; this is so in the theories of modular moonshine and the Jones polynomial, but not in the case of Donaldson/Floer theory. In the case of the Jones polynomial and Donaldson/Floer theory, the vector spaces $\mathscr{H}_X$ are finite dimensional, and a morphism of vector spaces is taken to mean an arbitrary linear transformation (preserving the unitary structure in the case of the Jones polynomial); in the theory of modular moonshine, the $\mathscr{H}_X$ are infinite dimensional, and it is necessary to be more precise about what is meant by a morphism among these spaces.

In Segal's language, the association $X \to \mathscr{H}_X$ is to be a functor from the category of d-dimensional manifolds with additional structure (and diffeomorphisms preserving the specified structures) to the category of vector spaces (and linear transformations of the appropriate kind).

Certain additional restrictions are imposed. The empty d-manifold $\varnothing$ is permitted, and one requires that $\mathscr{H}_\varnothing = \mathbb{C}$ ($\mathbb{C}$ here being a one-dimensional vector space with a preferred generator which we call "1"), If $X \coprod Y$ denotes the disjoint union of two d-dimensional manifolds X and Y, then one requires $\mathscr{H}_{X \coprod Y} = \mathscr{H}_X \otimes \mathscr{H}_Y$. If $-X$ is X with opposite orientation, and $*$ denotes the dual of a vector space, one requires $\mathscr{H}_{-X} = \mathscr{H}_X^*$.

Since the late 1920s, the spaces $\mathscr{H}_X$ have been known to physicists as the "physical Hilbert spaces" (of the particular quantum field theory under consideration). The association $X \to \mathscr{H}_X$ is roughly half of the basic structure considered in quantum field theory. The second half corresponds in physical terminology to the "transition amplitudes."

To introduce the transition amplitudes, we consider (Figure 2(a) on the next page) a cobordism of oriented (and possibly disconnected or empty) d-dimensional manifolds. Such a cobordism is defined by an oriented $(d+1)$-dimensional manifold W whose boundary is, say, $\partial W = X \cup (-Y)$, where X and Y are oriented d-dimensional manifolds (whose orientations respectively agree or disagree with that induced from W). It is required that whatever structure (conformal structure, framing, metric, etc.) has been introduced on X and Y is extended over W. Such a cobordism is regarded as a morphism from X to Y. To every such morphism of manifolds, a quantum field theory associates a morphism of vector spaces

$$\Phi_W : \mathscr{H}_X \to \mathscr{H}_Y. \tag{3.1}$$

Of course, this association $W \to \Phi_W$ should be natural, invariant under any diffeomorphism of W that preserves the relevant structures. Regarding $-W$ as a morphism from $-Y$ to $-X$, one requires that $\Phi_{(-W)} : \mathscr{H}_{(-Y)} \to \mathscr{H}_{(-X)}$ should be the dual linear transformation to Φ_W. And if $W = W_1 \cup W_2$ is a composition of cobordisms (Figure 2(b) on the next page), one requires that

$$\Phi_W = \Phi_{W_2} \circ \Phi_{W_1}. \tag{3.2}$$

These requirements correspond physically to relativity, locality, and causality.

A very important special case of this is the case in which W is a closed $D = (d+1)$-dimensional manifold without boundary. Such a W can be

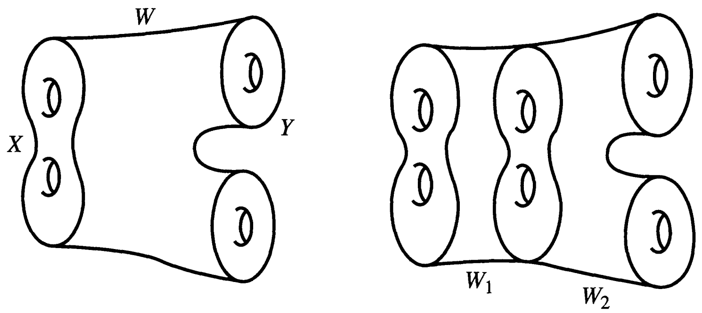

FIGURE 2. (a) A cobordism of oriented d-dimensional manifolds; (b) a composition of such cobordisms.

regarded as a morphism from the empty d-dimensional manifold $\varnothing$ to itself. Since $\mathcal{H}_\varnothing = \mathbb{C}$, the associated morphism $\Phi_W : \mathcal{H}_\varnothing \to \mathcal{H}_\varnothing$ is simply a number, which for physicists is often called the partition function of W and denoted $Z(W)$. This partition function is the fundamental invariant in quantum field theory; for different choices of theory, one gets the invariants of D-dimensional manifolds indicated in the last column in Table 1.

For $Z(W)$ to be defined in a given quantum field theory, W must of course be endowed with the structure appropriate to the particular quantum field theory in question. For instance, in the case of modular moonshine, W must be a Riemann surface with a conformal structure. In genus one, this means that W is an elliptic curve, which can be represented by a point in the upper half-plane subject to the action of the mapping class group. The naturality of the association $W \to Z(W)$ means that $Z(W)$ can depend only on the equivalence class of the conformal structure of W, and it is this which leads to modular forms. In our other two examples, no metric or conformal structure is present, so we are dealing with topological invariants. In our second example of the Jones polynomial, the invariant $Z(W)$ is an invariant of oriented three-manifolds which is an analog for three-manifolds of the Jones polynomials for knots in $\mathbf{S}^3$. (The actual knot invariants can be obtained by an elaboration of the quantum field theory structure.) In our third example of Donaldson theory, the invariant $Z(W)$ is the prototype of the invariants that appear in the celebrated Donaldson polynomials of oriented four-manifolds.

It is built into the axioms of quantum field theory that the fundamental invariants $Z(W)$ can be computed from a decomposition of the type that is known in the case of three-manifolds as a Heegaard splitting. This means a realization of W as $W = W_1 \cup W_2$, where W_1 and W_2 are D-manifolds joined together along their common boundary Σ. In this case the morphism

W from the empty manifold $\varnothing$ to itself factorizes as morphism W_2 from $\varnothing$ to Σ composed with a morphism W_1 from Σ to $\varnothing$, i.e.,

$$\Phi_W = \Phi_{W_1} \circ \Phi_{W_2}. \tag{3.3}$$

If 1 is the canonical generator of $\mathscr{H}_\varnothing$, we then have

$$Z(W) = (1, \Phi_W \cdot 1) = (1, \Phi_{W_1} \circ \Phi_{W_2} \cdot 1). \tag{3.4}$$

Let $v \in \mathscr{H}_\Sigma$ be the vector $v = \Phi_{W_2}(1)$. Also, think of $-W_1$ as a morphism from $\varnothing$ to $-\Sigma$, and let $w \in \mathscr{H}_{-\Sigma}$ be the vector $w = \Phi_{-W_1}(1)$. Then (3.4) amounts to

$$Z(W) = (w, v). \tag{3.5}$$

This ability to calculate via Heegaard splittings is part of the conventional *definition* of the Casson invariant (which has a quantum field theory interpretation analogous to that of Donaldson theory), and is essential in the calculability of the three-manifold invariants that are related to the Jones polynomial. Likewise, in the case of modular moonshine, the decomposition (3.5) is the key to the fact that the partition function $Z(W)$ can be written as the character $\mathrm{Tr}_{\mathscr{R}} q^H$ of equation (2.1).

4. Construction of quantum field theories. The question arises, of course, of how these quantum field theories are to be constructed. About this enormous subject it is possible only to say a few words here.

The starting point is always the choice of an appropriate *Lagrangian*, which is the integral of a local functional of appropriate fields. For instance, if one is interested in understanding the Jones polynomial, one picks a finite-dimensional compact simple group G and one considers a connection A on a G-bundle E over a three-manifold M. Let $F = dA + A \wedge A$ denote the curvature of this connection. On the Lie algebra $\mathscr{G}$ of a compact group G, there is an invariant quadratic form which we denote by the symbol Tr (that is, we write $(a, b) = \mathrm{Tr}(ab)$).[1] For the Lagrangian, we take the Chern-Simons invariant of the connection A:

$$\mathscr{L} = \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right). \tag{4.1}$$

(Here k is a positive integer, a fact that is required so that the argument $e^{i\mathscr{L}}$ in the Feynman path integral is gauge invariant.) To construct a quantum field theory from this Lagrangian, there are two basic requirements. First, we must construct a functor from Riemann surfaces Σ to Hilbert spaces $\mathscr{H}_\Sigma$; and second, for every cobordism W from Σ to Σ', we must construct a morphism $\Phi_W : \mathscr{H}_\Sigma \to \mathscr{H}_{\Sigma'}$.

[1] The quadratic form is to be normalized so that the characteristic class $\frac{1}{4\pi}\mathrm{Tr}\, F \wedge F$ has periods that are multiples of 2π.

For the first step, one proceeds as follows. Given the surface Σ, we consider the Lagrangian (4.1) on the three-manifold $\Sigma \times \mathbb{R}$. The space of critical points of the Lagrangian, up to gauge transformations, is known in classical mechanics as the "phase space" of the system under investigation. Let us call this phase space $\mathscr{M}_\Sigma$. In the case at hand, the Euler-Lagrange equation for a critical point of the Lagrangian (4.1) is the equation $F = 0$, where $F = dA + A \wedge A$ is the curvature of the connection A. (That is, (4.1) is invariant to first order under variations of the connection of compact support if and only if $F = 0$.) A flat connection on $\Sigma \times \mathbb{R}$ defines a homomorphism of the fundamental group $\pi_1(\Sigma \times \mathbb{R})$ into G. Of course, this is the same as a homomorphism of $\pi_1(\Sigma)$ into G. The classical phase space $\mathscr{M}_\Sigma$ associated with the Lagrangian (4.1) is simply the moduli space of homomorphisms of $\pi_1(\Sigma) \to G$, up to conjugation by G.

Now, it is a general fact in the calculus of variations that the phase space associated with a Lagrangian such as (4.1) is always endowed with a canonical symplectic structure ω. Indeed, this is how symplectic structures orginally appeared in classical mechanics, and as such it was the starting point of symplectic geometry as a mathematical subject also. In the case at hand, the symplectic structure thus obtained in $\mathscr{M}_\Sigma$ is known **[13]**, and has been studied very fruitfully from the point of view of two-dimensional gauge theory **[14]**, but my point is that this symplectic structure on $\mathscr{M}_\Sigma$ can be considered to arise from a *three-dimensional* variational problem. This elementary fact is an important starting point for understanding the mysterious three-dimensionality of the Jones polynomial.

Once the appropriate phase space $\mathscr{M}_\Sigma$ is identified, the association $\Sigma \to \mathscr{H}_\Sigma$ is made by "quantizing" $\mathscr{M}_\Sigma$ to obtain a Hilbert space $\mathscr{H}_\Sigma$. Geometric quantization is not sufficiently well developed to make quantization straightforward in general (or perhaps this is actually impossible in general), but in the case at hand quantization can be carried out by choosing a complex polarization of $\mathscr{M}$. This is accomplished by picking a complex structure J on Σ and using the Narasimhan-Seshadri theorem to identify $\mathscr{M}$ with the moduli space of stable holomorphic $G_\mathbb{C}$ bundles over Σ. This moduli space is then quantized by defining $\mathscr{H}_\Sigma$ to be the space of holomorphic sections of a certain line bundle over $\mathscr{M}$. This space is independent of J (up to a projective factor) because of its interpretation in terms of quantization of the underlying classical phase space $\mathscr{M}$. The association $\Sigma \to \mathscr{H}_\Sigma$ is the geometric origin of the Jones braid representations (or rather their analog for the mapping class group in genus g).

Once the association $\Sigma \to \mathscr{H}_\Sigma$ is understood, it remains to define for every cobordism W from Σ to Σ', a corresponding morphism $\Phi_W : \mathscr{H}_\Sigma \to \mathscr{H}_{\Sigma'}$. The key notion here is the "Feynman path integral," that is, Feynman's concept of integration over the whole function space $\mathscr{V}$ of connections on (the given G-bundle E over) W. Roughly speaking, the function space integral with prescribed boundary conditions gives the kernel of the morphism Φ_W.

I have tried to explain heuristically the role of functional integrals in **[12]**, and I will not repeat those observations here.

In conclusion, let me point out that if G is a *compact* group, then, as I have argued in **[15]**, the quantum field theory associated with the Lagrangian (4.1) is related to the Jones polynomial and its generalizations. However, (4.1) makes sense for any gauge group G with an invariant quadratic form "Tr" on the Lie algebra. It is natural to ask what mathematical constructions are related to the quantum field theories so obtained. One case that can be conveniently analyzed is the case in which one replaces the compact group G by a group $TG = \mathscr{G} \ltimes G$; here $\mathscr{G} \ltimes G$ denotes the semidirect product of G with its own Lie algebra $\mathscr{G}$, the latter regarded as an abelian group acted on by G. It turns out **[16]** that with this choice of gauge group, the quantum field theory derived from (4.1) (with a certain choice of "Tr") is related to recent work of D. Johnson on Reidemeister torsion **[17]**, while if instead one considers a certain super-group whose bosonic part is TG then (4.1) (again with a certain choice of "Tr") is related to the Casson invariant of three-manifolds. It is also very interesting to take G to be a semisimple but noncompact Lie group. The corresponding quantum field theories are very little understood, but there are indications that they should be very rich. In fact, it appears **[18]** that the theories based on $\mathrm{SL}(2, \mathbb{R})$ and especially $\mathrm{SL}(2, \mathbb{C})$ must be intimately connected with the theory of hyperbolic structures on three-manifolds, as surveyed the other day by Thurston.

5. Conclusion. In attending this meeting, I have found it striking how many of the lectures were concerned with questions that are associated with quantum field theory—and in many cases questions that might be characterized as questions about quantum field theory. In time we will hopefully gain a clearer picture of the scope of some of these newly emerging relations between geometry and physics. It is not too much to anticipate that many important constructions relating quantum field theory to topology and differential geometry remain to be discovered. Harder to foresee is whether—by the time of the one hundred fiftieth anniversary of the American Mathematical Society—the influence of quantum field theory will also extend to other areas of mathematics, such as algebraic geometry and number theory, which superficially might appear to be comparatively immune. Let me recall that in his lecture earlier this week, Dick Gross concluded by urging physicists and mathematicians to find a quantum field theory explanation for the appearance of modular forms in the study of the L-functions of elliptic curves. (This question was, of course, motivated by the relation of quantum field theory to the different kinds of modular moonshine.) Perhaps this challenge, or analogous ones about which one might speculate, will be met. Hints today concerning quantum field theoretic insights about number theory are probably no more compelling than hints of quantum field theoretic insight about differential geometry were ten years ago.

What significance might the emerging links between quantum field theory and geometry have for physics? It is very noticeable that the aspects of quantum field theory that are most useful in understanding the geometrical problems that I have been talking about are pretty close to the slightly specialized aspects of quantum field theory that appear in string theory. Modular invariance in the theory of affine Lie algebras is certainly a familiar story to string theorists. The Jones polynomial and its generalizations are related to the "rational conformal field theories" which are one of our main tools for finding exact classical solutions in string theory. The constructions that enter in formulating Donaldson theory as a quantum field theory are also very similar to what string theorists are accustomed to (in the use of world-sheet BRST operators).

Apart from being at least loosely connected with all of the geometrical problems that we have been discussing, string theory seems to be the center of some geometrical questions of central physical interest. The towering puzzle in contemporary theoretical physics is—at least from my standpoint—the puzzle of finding the geometrical context in which string theory should be properly formulated and understood. I am sure many physicists would share this judgment. With our present concepts, this problem (to which I attempted a thumbnail introduction in **[12]**) seems well out of reach. Perhaps it is not too far-fetched to hope that some insight in this mystery can be obtained from the further study of geometrical questions arising in quantum field theory.

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SCHOOL OF NATURAL SCIENCES, INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY 08540