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Instantons and Their Relatives

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Introduction and a glimpse of history. It was a great honor to be invited to give an hour talk at the Centennial Celebration of the American Mathematical Society. I have been reminded many times both before and after that I was the one woman speaker among the twenty odd speakers of this conference. Unlike some of my younger colleagues who gave addresses, I myself cannot expect to be present at the 150th anniversary celebration, although I very much hope a world sufficiently similar to ours exists in 2038 for such a celebration to take place. I also hope that no comments on the place of women in mathematics are even relevant at the time of this next celebration.

In preparing and writing up my talk, which was introduced by the legendary differential geometer S. S. Chern, I have been very aware that I am the only speaker representing the exciting developments which have taken place in global differential geometry in the last fifteen years. The technical understanding of elliptic partial differential equations has led to unprecedented understanding of the global aspects of diverse basic ideas in geometry such as minimal surfaces and Riemannian curvature equations. Applications in topology, algebraic geometry, and applied mathematics are very striking and important. My talk, however, concentrated on a completely new subject: the study of curvature or field equations linked not to the geometry of the manifold but to the extrinsic geometry of objects with the technically obscure name of principal bundles. Their structure groups appear in particle physics as the SU(2) is isospin; the SU(3)'s of isospin and strangeness, color, and charm; the SU(5) of unified field theories; and the E_8 's of string theory.The classical equations of the gauge theory of theoretical physicists entered pure mathematics. In a very few years they have become central to mathematicians' understanding of objects such as smooth four-manifolds and stable bundles. This development has been the event of greatest intellectual excitement in my career. I am left with the feeling that the few small contributions I made in gauge field theory were right in the center of intellectual progress.

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Hence my continued pursuit of a chimera: an understanding of mathematical ideas as they come from outside mathematics itself. My energies have taken me only to the mathematical edge of physics.

Before I begin the official write-up of my talk, I would like to take this opportunity to thank all the people and institutions who have encouraged and supported my mathematical career thus far. Mention of a special sort should go to my thesis advisor, Richard Palais, and to my good friend, S. T. Yau. Finally, my junior colleague, Dan Freed, has been of great help in preparing this manuscript.

1. The birth of gauge theory. How did gauge theory appear and become successful in mathematics in the space of a few years? The fundamental mathematical ingredients were in place. The basics of fibre and vector bundles and their connections were in daily use by geometers. Chern-Weil theory (and even Chern-Simons invariants) were studied in most graduate courses in differential geometry. De Rham cohomology and its realization via the Hodge theory of harmonic forms were standard items in differential topology. In hindsight, the Yang-Mills equations were waiting to be discovered. Yet mathematicians were in themselves unable to create them. Gauge field theory is an adopted child.

Physicists Yang and Mills wrote down their equations in 1954, referring explicitly to isotopic spin as the group invariance. Some ingredients of gauge theory became incorporated gradually into the theory of the electro-weak interactions in physics over the next twenty years. These are not particularly recognizable or striking to mathematicians as gauge theories because of their "broken symmetry." There were isolated cases of mathematicians noticing the importance of these equations, but no essential impression was made on the mathematical community as a whole.

The original Yang-Mills equations are nonlinear extensions of Maxwell's equations in space-time. This means they are a system of second-order partial differential equations in the four variables of space and time ($3 + 1$ dimensions). In the process of studying the quantum theory, the solutions to the Euclidean four-dimensional second-order equations become important. A series of important papers in the seventies starts with the discovery of the first-order self-dual equations and the single instanton solution (which one can think of as rotationally symmetric in $\mathbb{R}^4$) by the Russian physicists Belavin, Polyakov, Schwarz, and Tyupkin in 1975. Conformal invariance produces a five-dimensional family, which has turned out to be a complete family of solutions of energy $8\pi^2$. A larger $5|k|$ parameter family of energy $|k|8\pi^2$ for all k was almost immediately discovered by a number of physicists (Wilcek; Corrigan and Fairlie; Jackiw, Nohl, and Rebbi). The form of the solutions employed the "'t Hooft Ansatz" of reducing the equations to the linear equation $\Delta\varphi = 0$. Almost immediately, mathematicians were able to contribute information on solutions using an amazing variety of techniques from modern mathematics. Many of the original ideas were due to Michael Atiyah, and in those early years he lectured all over the world on instantons and gauge theories. Much of the purely mathematical development is surely due to the interest and excitement he conveyed to his mathematical audiences at that time.

I have left the rest of the early mathematical development to be inferred from the table of papers in the subject. Simon Donaldson received the Field's Medal in 1986 primarily for the work in his Ph.D. thesis, published in 1983 in the Journal of Differential Geometry. Gauge theory has proved itself an important tool of mathematics, which I for one believe will last. The following list of Early Papers in Gauge Theory and the bibliography are but a small part of the evidence. Mathematical gauge theory provides the best understood invariants for topological 4-manifolds [**D1**, **A1**], more invariants for homology 3-spheres [**A2**], a description of the moduli space of stable holomorphic bundles over Kähler manifolds [**D2**], a tool for uniformization theorems [**S**], concrete descriptions of cosmological objects, as well as a special model toy for sophisticated mathematicians investigating abstract mathematical phenomena.

It is an important part of physics? Has the child adopted by mathematicians been rejected by its natural parents? The physics papers in 1975–77 listed in the following table are in general part of a scheme to describe quantum chromodynamics. They represent failed attempts to understand strong interactions by "tunnelling effects." The physics behind these equations has in general been completely mysterious to mathematicians, who are continually frustrated in their attempts to either understand or believe even the simplest calculations in quantum field theories of this geometric sophistication. The failure of this model does not mean they are not present in physics. In their broken form they are used in calculation for the standard model of particles. Lattice gauge theory is a thriving user of CRAY time. String theory interactions pre-suppose quantized gauge theories. Those who study the physics referred to in the talks of Vaughn Jones, Victor Kac, and Ed Witten will find them very much behind the group representation theory which has become so important. So far we mathematicians have been able to make use only of the classical theory which was not of much use in theoretical physics. Good geometric mathematical models for quantizing gauge field theories promise to be interesting to both sets of parents.

Early Papers in Gauge Theory

(1954) C. N. Yang and R. Mills, *Conservation of isotopic spin and isotopic gauge invariance*, Phys. Rev. **96**, 1–9.

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(1976) G. 't Hooft, *Computation of the quantum effects due to a four-dimensional pseudoparticle*, Phys. Rev. D **14**, 3432–3450.

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(1983) S. Donaldson, *An application of gauge theory to the topology of four-manifolds*, J. Differential Geom. **18**, 269–316.

2. What is an instanton? What is all the excitement about? What is an instanton, and why is it important?

Instantons are 4-dimensional objects, and share a lot of properties with vortices (two-dimensional objects used to describe superconductivity), monopoles (three-dimensional cosmological objects), and Hermitian Yang-Mills metrics (complex objects of any complex dimension). These are all close relatives in a large diverse family of global geometric mathematical objects defined by partial differential equations. We can include in this extended family objects like geodesics, minimal and constant curvature surfaces, and black holes. All these objects extend simple well-understood physical models, satisfy nonlinear equations obtained from variational principles, exhibit certain topological properties, have a gauge or coordinate invariance, and have important modern applications in topology, algebraic geometry, and applications. There are many good descriptions of the basic equations of gauge theory in the list of basic reference books in §4. Here we give a more impressionistic view by comparing the characteristic properties and behavior of instantons with similar phenomena exhibited by minimal surfaces. It is a powerful mathematical fact that intuition and techniques can be passed back and forth between the two.

Physical origins. The minimal area surface equation was written down by the Belgian physicist Plateau as an equation satisfied approximately by soap films. Since nearly all of us played with soap films and bubbles as young or not so young children, we think we understand minimal area surfaces conceptually. One imagines the surface in the familiar three-space we live in, and then transposes it into a curved space created by the imagination.

Instantons represent tunnelling from flat three-dimensional space to itself. No one plays with quantum effects as a child, and even the geometric connection or vector potential which replaces the concept of surface is hard to imagine. I think of the vector potential as having some of the stretchy properties of surfaces, but sitting over the spacial manifold, not in it.

Variational formulation. The concept of least area is familiar to me from simple calculus problems. However, the formula for area of a general surface in a three-manifold is quite complicated. The minimal area principle is often replaced by minimal energy, or the L^2 norm of the derivative of the embedding, $s: \Sigma \to X$.

$$E(s) = \int_\Sigma |ds|^2 (du)^d.$$

The Yang-Mills integral is simply the L^2 norm of the curvature (or field):

$$\mathrm{YM}(A) = \int_X |F_A|^2 (du)^d.$$

For $\dim \Sigma = 2 = d$ and $\dim X = 4 = d$, both integrals are conformal invariants. This allows one to compactify $\Sigma = \mathbb{R}^2 \cup \{\infty\} = S^2$ and $X = \mathbb{R}^4 \cup \{\infty\} = S^4$ in exactly the same way, and produces the same borderline behavior for Morse theory. We think of the lack of compactness in Yang-Mills at points much like we think of the "bubbling" of minimal surfaces. Both are caused by scale or conformal invariance **[SU, Se]**.

Linear models. Both the minimal surface (or harmonic map equations) and the Yang-Mills equations can be thought of as nonlinear generalizations of Hodge-de Rham theory. Harmonic p-forms $\alpha \in \Omega^p(M, \mathbb{R})$ satisfy the closed condition $d\alpha = 0$ and the Hodge equation $d * \alpha = 0$.

If $s: M \to N$ is a map, then $ds = \alpha$ is a one-form with values in s^*TN. In this context $d_s\alpha = 0$ is an identity and $d_s * \alpha = 0$ is the harmonic equation. Likewise, if F_A is the curvature of a connection A, $D_A F_A = 0$ is the Bianchi identity and $D_A * F_A = 0$ is the Yang-Mills equation.

First-order equations. Both the harmonic map equation and Yang-Mills are d-dimensional equations, where d is arbitrary. However, in the scale invariant case, we have special equations. For $\alpha = ds$, a one-form, and X a complex Kähler manifold with complex structure operator J, a special very tractible class of minimal surfaces or harmonic maps are the holomorphic or antiholomorphic ones. They satisfy the Cauchy-Riemannian equations

$$J(s)\alpha = \pm * \alpha.$$

Likewise, instantons and anti-instantons satisfy a similar first-order equation:

$$F_A = \pm * F_A.$$

Complex equations. For Σ and X arbitrary complex manifolds, important examples of harmonic maps (or minimal surfaces) are generated by holomorphic maps $\Sigma \to M$. For Yang-Mills, if X is complex, there is a special form of the Yang-Mills equations which requires the curvature F_A to be a two-form of type $(1, 1)$ which is traceless with respect to the Kähler form.

Gauge invariance. The concept of a minimal surface does not carry with it a preferred coordinate chart. Jesse Douglas' original solution to the Plateau problem (for which he received half the first Field's medal) uses conformal coordinates obtained via the Riemann mapping theorem and replaces area by energy [**Do**]. No such elegant global solution for gauge fixing has emerged for the coordinate problem in gauge theory. However, locally on the manifold or locally in the space of connections, harmonic slices are used for technical constructions.

Topological applications. Minimal surfaces can be used to study the topology of three-manifolds [**MY**]. The solutions to the instanton equation have emerged as the main tool for studying the topology of differential four-manifolds [**A1**].

Moduli spaces. In studying moduli spaces of minimal surfaces, one is forced to look to the Riemann moduli space for the models. Moduli spaces of solutions to Yang-Mills exhibit many similarities with these same model spaces of complex structures on Riemann surfaces. Similar compactification phenomena exist.

Examples. One of the most satisfying aspects of the study of minimal surfaces in 3-space is the existence of many immediate examples via the Weierstrass representation. If f and g are any two holomorphic functions on Ω, then the piece of surface

$$\Sigma = \{\mathrm{Re}(f(z)(1 - g^2(z)), if(z)(1 + g^2(z)), 2f(z)g(z)), \ z \in \Omega\} \in \mathbb{R}^3$$

is a minimal surface. This, and some hard to come by expertise with computer graphics are all that one needs to draw many beautiful pictures of minimal surfaces [**H**].

The Penrose transform converts solutions of $F_A = - * F_A$ to holomorphic bundles over $\mathbb{CP}^3$. However, the complex analysis is considerably more complicated! Fortunately, the 't Hooft Ansatz produces solutions $A = \mathrm{Im}(\frac{\partial}{\partial q}(\ln \varphi) dq)$ to Yang-Mills from solutions to $\Delta\varphi + \lambda\varphi^3 = 0$ on $\mathbb{R}^4$, if we regard $\mathbb{R}^4 = \mathbb{H}$ as the quaternions. Some simple examples are available, although I do not know what computer pictures would look like. I have seen some elegant pictures of vortices and monopoles however [**HMRVW**].

3. Abelian vortices. The importance of the class of equations found in gauge theory lies almost entirely in the structure of their moduli spaces of solutions. Of course, the moduli space of solutions to the self-dual Yang-Mills equations on a four-manifold X is very complicated. However, in some cases, special simpler solutions can be constructed using symmetries **[T1]**. Here we present these simpler equations and describe the moduli space of solutions on a Riemann surface. I feel this is appropriate. Clifford Taubes' description in his thesis of the solutions to the vortex equation on $\mathbb{R}^2$ was one of the first analytical results on moduli spaces in gauge field theory **[T2]**. The present result is contained in a 1988 Ph.D. thesis of a Ph.D. student of mine, Steven Bradlow **[B]**.

In this application, we let X be a Kähler manifold (it will specialize at the end to a Riemann surface). The integral for the coupled Yang-Mills-Higgs equations has the general form

$$\int_X \left[|F_A|^2 + |D_A\Phi|^2 - \frac{\lambda}{4}(t - |\Phi|^2)^2\right] (du)^d.$$

Here F_A is the curvature in a principal bundle and Φ is a section of an associated bundle. The parameters λ and t are real. This integrand is particularly easy to describe if the group is $U(1)$ and Φ is a section of a line bundle E. Fix a base connection D_0 on E. The unknowns are a complex-valued functions Φ twisted to lie in E and an ordinary one-form A,

$$F_A = F_0 + d(iA), \qquad D_A\Phi = D_0\Phi + iA \cdot \Phi.$$

The Euler-Lagrange equations are second order and have the form

$$d * F_A + \operatorname{Im}\langle *D_A\Phi, \Phi\rangle = 0, \qquad \Delta_A\Phi + \frac{\lambda}{2}(-t + |\Phi|^2)\Phi = 0.$$

One family of solutions can be obtained by considering the solutions

$$\Phi = 0, \qquad d * F_A = 0.$$

These are the usual Yang-Mills equations and linear Hodge-de Rham theory provides an analysis of these.

However, on a Kähler manifold we can use the complex structure to integrate the functional by parts to obtain an equivalent integral

$$\int_X \left[|H_A|^2 + 4|F_A^{0,2}|^2 + 2|\bar\partial_A\Phi|^2 - H_A|\Phi|^2 + \frac{\lambda}{4}(t - |\Phi|^2)^2\right] (du)^d - 8\pi^2 \mathrm{ch}_2 E.$$

Here $F_A^{0,2}$ is the $(2, 0)$ part of the curvature, $H_A = (\omega, F_A)$ is the contraction of the curvature two-form with the Kähler form ($*F_A$ in two real dimensions), and $\mathrm{ch}_i E$ indicate topological contributions from the Chern-Weil formulas. The first-order equations come from this integration by parts exactly as they do for Yang-Mills or monopoles.

THEOREM. *For $\lambda = 1$, we have that the Yang-Mills-Higgs integral is bounded below by the number $-8\pi^2\mathrm{ch}_2(E) + 2\pi t\,\mathrm{ch}_1 E$. This topological minimum is taken on by solutions to the first-order equations $\bar\partial_A\Phi = 0$, $F_A^{0,2} = 0$, and $2H_A - |\Phi|^2 + t = 0$.*

The surprise is that it is completely straightforward (given a little standard complex differential geometry) to find all the solutions of this equation.

THEOREM (Bradlow). *For fixed* $t > t_0 = 4\pi \mathrm{ch}_1 E(\mathrm{vol}\, X)^{-1}$, *the solutions of the first-order equations correspond in a one-to-one fashion to holomorphic sections of holomorphic line bundles.*

This gives a really simple description of the moduli space of solutions over a Riemann surface Σ. For $\mathrm{ch}_1 E = k$, specifying the k zeros of a holomorphic section on Σ gives both the section and the line bundle.

COROLLARY. *The moduli space of solutions to the first-order vortex equations on a Riemann surface in a bundle with* $\mathrm{ch}_1 E = k$ *corresponds to the space* $\mathcal{S}^k(\Sigma)$ *of* k *unordered points on* Σ, *possibly with multiplicity. These points correspond to zeros of the Higgs field in the solution.*

The analysis in this sample example reduces to the solution of an elliptic equation of the form

$$-\Delta u + |\Phi|^2 e^u - (t - t_0) = 0$$

for a change of metric e^u in the bundle E. This equation was studied by Kazdan and Warner in conjunction with their investigation of conformal deformations of metrics in two dimensions. Of course, the interesting and new nonlinear analysis is in the extensions to the nonabelian case, where contact is made with notions of stability in algebraic geometry. This work is also in Bradlow's thesis **[B]**. However, this simple abelian example serves to illustrate what we can expect moduli spaces to look like.

4. And if there is a twenty-first century.... I am a pessimist. If I think about the future, I think mainly about overpopulation, AIDS, fiscal instability, the threat of nuclear war, and myriads of different seemingly unsolvable social and environmental problems. In making my predictions, I must confess that I worry there will be no twenty-first century suitable for the pursuit of mathematics.

However, I am a mathematical optimist. It seems to me that Mathematics is intellectually in great shape. Current developments are exciting. The problems of the world are not reflected as problems within the world of mathematics. This is certainly one of its attractions for me. But I see it as more than a refuge from real life. To me real progress has been made in mathematics in the twenty years I have been a member of the community, wheras I am not so sure about progress in the real world. During these years the world of mathematics has opened up to make contact with neighboring intellectual disciplines. We have been influenced by our old friends from theoretical physics—but even more by greater changes such as the advent of the computer age and by the daily use of mathematics in technology. The response within the discipline of mathematics has been very positive, if a bit slow and conservative. As a result, the content of mathematics in the form of its fundamental ideas seems to me to be much richer. I look forward to the next decades in the development of mathematics with great curiosity and hope.

PREDICTION 1. *Simplicity through complexity.*

There are two basic approaches to simplicity in mathematics. One approach struggles with the choice of description of the mathematical object via bases or coordinates until one obtains the right and self-evident minimalist description. On the other hand one can throw in all possible descriptions (as my mother used to say, everything and the kitchen sink) and then divide out by equivalences to obtain a simple classification of objects. This suits today's complex world. The success of gauge theory is via this second complexification route. I think we are not done with this trend of enlarging the class of mathematical objects to unreal proportions and then dividing out by even larger equivalence classes. One sees this scheme working in the successful BRS quantization methods of physicists, and I predict we will continue to follow this pattern, at least in geometry [**FGZ**].

PREDICTION 2. *More beyond partial differential equations.*

The last fifteen years has seen the domination of differential geometry by techniques from partial differential equations. This might appropriately be called the "Yau School" of differential geometry [**Y**]. If the influences from physics continue, I think the era of domination will end. We should ask ourselves, "What will the discovery of a unified field theory in theoretical physics mean for differential geometry? The goal of a unified field theory is to meld the theory of gravity (geometry) with particles (groups). Geometry via algebra? It is not a completely new idea, of course.

PREDICTION 3. *Geometric understanding of quantum field theory.*

I hope that we mathematicians will soon have done with our fundamental difficulties with quantum theory. We have had fifty years of lack of success in explaining why Feynman diagrams work. I agree with many other speakers that we will soon decode the complexity of conformal field theories and redefine them as basic and simple mathematical objects. Topological quantum field theory holds out a really hopeful new possibility for axiomatic approaches to quantum field theory which will capture the essential geometric ideas for mathematicians without bogging down in analytical contradictions.

PREDICTION 4. *And after theoretical physics* ?

We are going through a period where the primary outside influence on differential geometry has been either cosmology or fundamental physics. We learn the importance of 1, 2, 3, 4, 10, 26, and ∞ dimensions (i.e., we study small and infinite-dimensional manifolds). We should remember there are other sources of inspiration. How about robotics, which must be done in large but finite dimensions and which is too complicated to be exact?

PREDICTION 5. *Return to Boubaki.*

My last comment is on style. I was generally taught in the famous Boubakist style of definition, theorem, proof, and maybe example if there is time. Coordinate descriptions were out: abstraction was in. Generalization abounded. If you cannot do it in n dimensions, do not bother. This approach dates back at least to Hilbert and his famous problems. My own mathematical interests and the predominant mathematical style today has become far more oriented towards the particular. One can also identify this trend in the talks in this Centennial Celebration. Coordinate descriptions are a universal language. Low dimensional topology is central. Lots of us think by example. I do consider it unfortunate that examples are basically the only thing in many useful subjects in mathematics which I personally do understand. This evolution to the particular has been an important part of the opening up and reaching out of the last twenty years.

Many scientists complain to me of the mathematical style of teaching of twenty years ago. The outside world has barely yet understood the change! I think this change has evolved along with the practical experiences of our universal teaching experience. It is here the world may have had its effect on mathematics: through our calculus students.

I see signs of reversion to the general in the next generation of students. They again think coordinates, low dimensions, and SU(2) are old-fashioned. This is as it should be. They have their own mathematics and its style to discover.

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