Proceedings of the AMS Centennial Symposium
August 8–12, 1988

Bounds, Quadratic Differentials, and Renormalization Conjectures

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Introduction. Consider mixing a deck of n cards by shuffling as usual after turning over one of the stacks. The resulting permutations are building blocks of the rich dynamics of mappings which fold the line. More specifically, given such a shuffle permutation σ which is irreducible there is a folding mapping f of an interval I (Figure 1(b)) and a smaller interval $I_1 \subset I$ about the turning point of f whose inverse orbit under f contains a *disjoint finite* collection of intervals permuted according to σ under the iteration of f. In fact it is known that each such shuffle permutation of intervals happens for some member of any complete family of mappings as indicated in Figure 1(b) (Milnor and Thurston **[MT]**, Sharkovski **[Sh]**).

We make use of the operator $f \to f_1$, where f_1 is the first return mapping of f to I_1. *This* $f_1 = (f^n/I_1)$ *is again a folding mapping of an interval* and $f \overset{R}{\mapsto} f_1$ is called *renormalization.* The operator R is partially defined on folding mappings and is specified precisely by taking n minimal and I_1 minimal. If R is defined and continues to be defined we define $f_1 = Rf$, $f_2 = Rf_1, \dots, f_n = Rf_{n-1}, \dots$. *We say f is infinitely renormalizable of type $(\sigma_0, \sigma_1, \sigma_2, \dots)$, where the σ_i are the shuffle permutations that arise inductively.* Infinitely renormalizable mappings of every type $(\sigma_0, \sigma_1, \sigma_2, \dots)$ occur in every complete family (Figure 1(b)).

The description renormalization comes from statistical physics. An analogy was discovered between phenomena there like critical opalescence and one of the examples here, type $(\tau, \tau, \tau, \dots)$, where τ is the permutation of order 2 (see Figure 4 from **[CT2]** on page 419). This type appears at the end of a cascade of period doubling bifurcations in any complete family. The physicists Feigenbaum **[F1]** and Coullet and Tresser **[CT1]** working numerically, and independently, in the U. S. and France found universal numerical characteristics about this cascade and about the limit geometry of

1991 *Mathematics Subject Classification.* Primary 58F13, 58F14; Secondary 30D05, 30F60, 39B12.

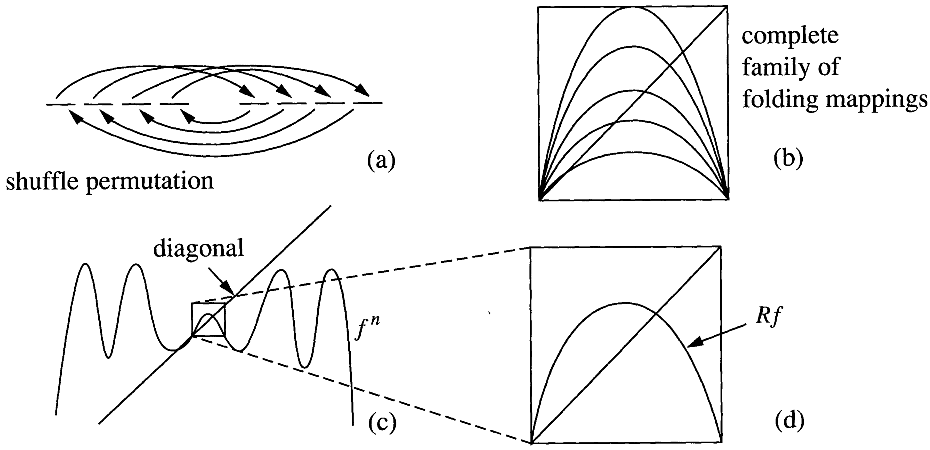


FIGURE 1

the type $(\tau, \tau, \dots)$ orbital Cantor set. *For example, in terms of any smooth control parameter the bifurcations occurred faster and faster at a relative rate* $(4.6692\dots)^n$, *n large. The boxes within boxes for the renormalization of type* $(\tau, \tau, \tau, \dots)$ *had the limiting successive ratio* $0.3995\dots$. *Finally, the graph inside the tiny box became canonical.* These characteristics were *universal* in the sense that they were computed numerically to be *independent of the choice of complete family of mappings with quadratic turnings.*

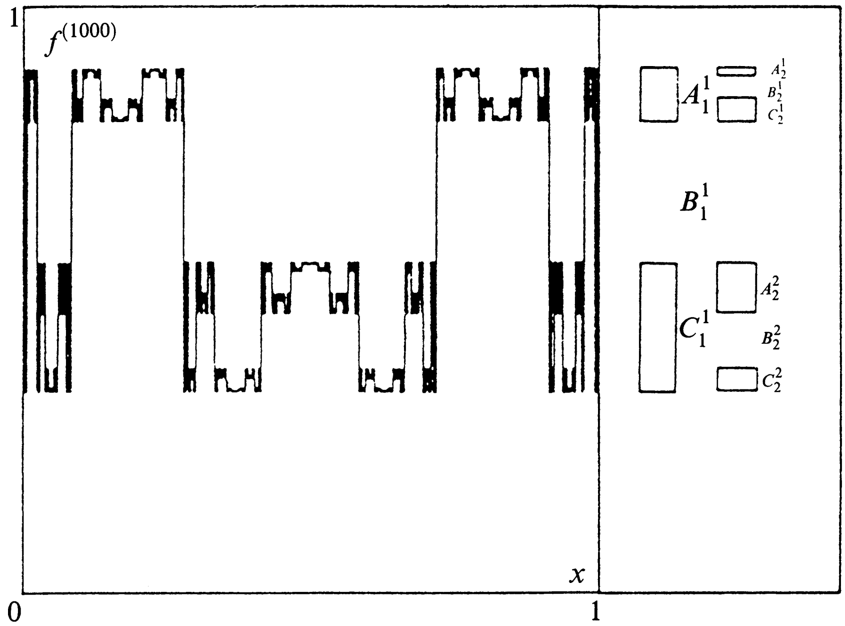
Some of us have been wondering for a long time what the *domain of validity of these discoveries* is and what *techniques from dynamics* need to be employed or invented to make a proof. We describe results obtained over the last few years in the following theorems.

Let us consider continuous mappings $f: I \to I$, where $I = [a, b]$, $f(a) = f(b) = a$, and f is a local homeomorphism except at one turning point c. To express the smoothness we require, write $f = Qh$, where $Q: I \to I$ is a quadratic polynomial and $h: I \to I$ is a homeomorphism. This decomposition is unique. We say f is a *smooth quadratic-like mapping bounded by* B if h is a diffeomorphism and $\varphi = \log|h'|$ satisfies the Zygmund and $\frac{1}{2}$-Hölder inequalities,

$$(*) \qquad \left|\varphi\left(\frac{x+y}{2}\right) - \frac{\varphi x + \varphi y}{2}\right| \le B|x-y| \quad \text{and} \quad |\varphi x - \varphi y|^2 \le B|x-y|.$$

Thus, all $f = Qh$, where h is a C^2 diffeomorphism, are included but our technique does not prove anything for smoothness lower than $(*)$ which implies $\log h'$ has a $t \log t$ modulus of continuity (see §1). We *note* that the h satisfying $(*)$ are precompact in C^1 diffeomorphisms and any limit satisfies $(*)$.

The first theorem uses combinatorics, real analysis (§1), and the de Melo-Van Strien **[MV1]**, Swiatek **[Sw]** Koebe distortion technique in real dynamics



[CT2, Figure 4] $f_R^{(1000)}(X)$ for $R = R_c$ is plotted on the left part. One sees the Cantor-like structure of the asymptotic orbit. On the right we sketch the first steps of the usual geometrical construction of this Cantor set.

extended to $(*)$ in §2. Theorem 1 is also valid for $|x|^r$ singularities, $r > 1$.

THEOREM 1 ("beau" property of renormalization). *If f is a smooth quadratic-like mapping bounded by B, then all the renormalizations $f_1, f_2, \ldots$ are smooth quadratic-like mappings with a bound depending only on B. After a number of renormalizations only depending on B further renormalizations are bounded universally.*

Using Theorem 1, the note, and the fact the critical value stays away from a (§4), we can form renormalization limits.

If $f_0 = \lim_{i\to\infty} R^{n_i} f$, set $f_1 = \lim'_{i\to\infty} R^{n_i - 1} f$, $f_2 = \lim'_{i\to\infty} R^{n_i-2} f$, etc., where $\lim'$ means we take limits over subsequences. Whenever $f_1, f_2, \ldots$ are in the domain of renormalization (which is assured if we assume the individual renormalization return times are uniformly bounded), then we obtain an *inverse chain of limits related by renormalization,*

$$(**) \qquad \cdots \to f_{n+1} \xrightarrow{R} f_n \to \cdots \to f_2 \xrightarrow{R} f_1 \xrightarrow{R} f_0,$$

where each f_n is a smooth quadratic-like mapping bounded by some universal B.

Denote the type of f_n by $\tilde{\sigma}_n = (\sigma_0^n, \sigma_1^n, \ldots)$ so that $\tilde{\sigma}_n = (\tilde{\sigma}_{n+1}$ shifted by one). Let $\tilde{\sigma} = \varprojlim \tilde{\sigma}_n$ be the inverse limit 2-sided infinite sequence

$\tilde{\sigma} = (\dots, \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \dots)$.

The second theorem, whose explicit formulation was motivated by a lecture of Curt McMullen at IHES (May 1990), uses holomorphic dynamics and quasiconformal mappings. It is valid also for analytic singularities x^{2k}, $k = 1, 2, \dots$, but remains mysterious for $|x|^r$ singularities, r real and greater than one. W. Paluba **[Pa]** has recently made some progress for r real (see also **[E2]** and **[CEL]**).

THEOREM 2 (Generalized Feigenbaum functions). *For each bi-infinite sequence $\tilde{\sigma} = (\dots, \sigma_{-1}, \sigma_0, \sigma_2, \sigma_2, \dots)$ of uniformly bounded shuffles there is one and only one analytic function $F(\tilde{\sigma})$ which is a smooth quadratic-like mapping, infinitely renormalizable of type $(\sigma_0, \sigma_1, \sigma_2, \dots)$, and which lies at the beginning of an infinite inverse chain of uniformly bounded smooth quadratic-like mappings related by renormalization of combinatorics $(\dots, \sigma_{-2}, \sigma_{-1})$,*

$$\cdots \to f_n \xrightarrow{R} f_{n-1} \to \cdots \to f_2 \to f_1 \to F(\tilde{\sigma}).$$

COROLLARY 1. *For each shuffle permutation σ and any smooth quadratic-like f (i.e., $f = Qh$, $\log h'$ Zygmund) of type $(\sigma, \sigma, \dots)$ we have*

$$\lim_{n\to\infty} R_\sigma^n f = F(\dots, \sigma, \sigma, \sigma, \dots).$$

Here R_σ means the renormalization associated to the shuffle σ. For example, the stable manifold of the Feigenbaum renormalization operator R_τ at the fixed point $F(\dots, \tau, \tau, \tau, \dots)$ consists of all the smooth quadratic-like mappings of type $(\tau, \tau, \tau, \dots)$.

If $f: I \to I$ is infinitely renormalizable of type $(\sigma_0, \sigma_1, \dots)$, then I contains n_0 disjoint intervals on each of which a conjugate of $f_1 = R(\sigma_0)f$ is defined. Each of these n_0 intervals contains n_1 intervals on which a conjugate of $f_2 = R(\sigma_1)R(\sigma_0)f$ is defined, etc. These interval collections nest down to a closed set which is the closure of the critical point orbit of f. We see in §3 that the total length of these intervals tends to zero exponentially quickly in the depth of renormalization, in the case of smooth quadratic-like mappings. Also, in the bounded type case (degree σ_i bounded), these Cantor sets have *bounded geometry* in the sense that the ratio $r_{\alpha\beta}$ between the length of an interval I_α at one level n and the length of an interval or gap β in I_α at the next level $n+1$ has a bounded logarithm. Actually more is true, *for bounded combinatorics.*

MAIN THEOREM (Coullet-Tresser geometric rigidity of Cantor sets). *All smooth quadratic-like mappings of type $(\sigma_0, \sigma_1, \dots)$ have critical orbit Cantor sets with the same universal ratio asymptotics. Namely, if f and g have type $(\sigma_0, \sigma_1, \dots)$, then uniformly in the depth n*

$$\lim_{n\to\infty} (r_{\alpha_n\beta_n}(f) - r_{\alpha_n\beta_n}(g)) = 0.$$

For example, the computed self-similarity ratio $0.3995\ldots$ *of successive intervals nesting down to the critical point in the period doubling examples appears in the Cantor set of any smooth quadratic-like mapping of type* $(\tau, \tau, \ldots)$. *Also, the Hausdorff dimension of the Cantor set of any such* f *is* $0.538045\ldots$.

This theorem, stating that quasiperiodic orbital Cantor sets of bounded type $(n_0, n_1, \ldots)$ have a rigid geometric structure at asymptotically fine scales, was our main objective (cf. **[DGP]**). In Rand **[R2]** and Sullivan **[S3]** independent proofs were given that the rigidity of the $(\tau, \tau, \tau, \ldots)$ Cantor set follows from and implies convergence of $R_\tau^n f$ to a universal limit. The study here of renormalization naturally leads to a stronger theorem about the structure of renormalization.

Let $\cdots \to f_n \xrightarrow{R} f_{n-1} \to \cdots \to f_2 \to f_1 \to f_0$ be any inverse chain of bounded smooth quadratic-like mappings related by renormalizations, $R(\sigma_n)[f_n] = f_{n-1}$, where degree σ_n is bounded.

THEOREM $2'$. (a) *The mappings* f_n *are canonical analytic functions determined by the combinatorics* σ_i *and a real number* $c \in [-2, 1/4]$. *The real number* c *is determined by the complex analytic extension of* f_0 *which is complex quadratic-like in the sense of Douady-Hubbard. The real number* c *is the unique element of* $[-2, 1/4]$ *so that* f_0 *is qc conjugate to* $z \to z^2 + c$ *on a neighborhood of the invariant set by a conjugacy which is a.e. conformal there* **[DH1]**.

(b) *For any bounded combinatorics and any* $c \in [-2, 1/4]$ *there is an inverse chain* (*unique by* (a)) *with these invariants.*

Theorems 1, 2, and $2'$ describe the dynamics of bounded time renormalization on smooth quadratic-like mappings.

(i) *A folding mapping is either finitely renormalizable or it is infinitely renormalizable and under repeated renormalization it becomes universally bounded.*

(ii) *Any smooth quadratic-like mapping is in the image of each renormalization* $R(\sigma)$, *but only the canonical mappings of Theorem* $2'$ *are in the infinite image of renormalization restricted to a bounded part of the space of folding mappings.*

(iii) *We see topologically a hyperbolic set for renormalization with points labeled by the bi-infinite combinatorics* $(\ldots, \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2, \ldots)$, *unstable manifolds labeled by backwards combinatorics* $(\ldots, \sigma_{-2}, \sigma_{-1}, \ldots)$ *and canonically parametrized by the Douady-Hubbard internal class* c, *and finally stable manifolds labelled by the foward combinatorics* $(\sigma_0, \sigma_1, \ldots)$ *and consisting of all the infinitely renormalizable mappings of type* $(\sigma_0, \sigma_1, \ldots)$ (*see Figure* 2 *on next page*).

Along the way to Theorem $2'$ we derive information about the complex analytic structure of renormalization limits. Say a folding mapping f has the *Epstein form* if $f = hQ$ and h^{-1} has a complex analytic injective (in

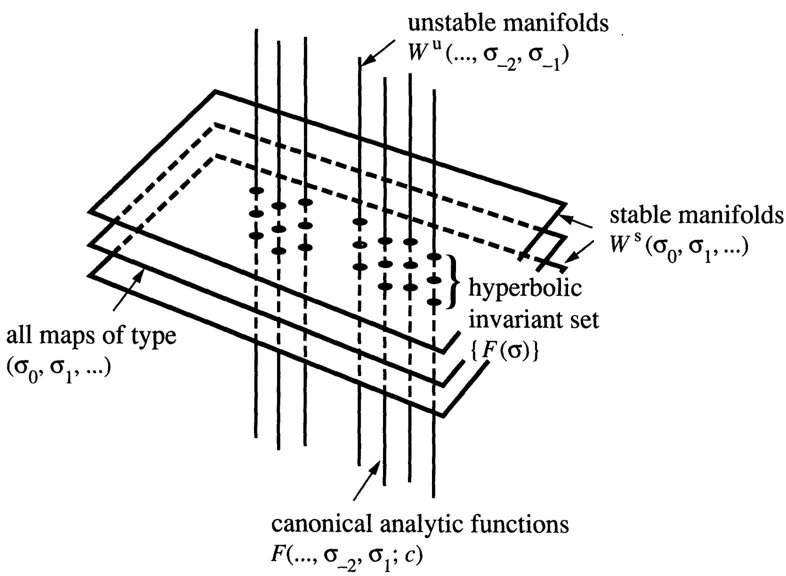


FIGURE 2

short "schlicht") extension to $\mathbb{C} - \{x$ real outside an open neighborhood of $I\}$. We write $f \in E(\lambda)$ if the open neighborhood of I is $\{x$ so that distance $(x, I) \leq \lambda$ length $I\}$.

THEOREM 3. *There is a universal $\lambda > 1$ so that if f is a smooth quadratic-like mapping bounded by B, then any limit of renormalization,* $\lim R^{k_n} f = F$, $k_n \to \infty$, *has the Epstein form and belongs to $E(\lambda)$. Unbounded combinatorics are allowed in this statement.*

Theorem 3 is based on the analytic estimates of the disjoint interval collections at the nth renormalization level. As we mentioned, the sum of lengths goes to zero exponentially in n. Also, the sum of the squares of the integrals of $|dx/x|$ over all the intervals except the one containing the critical point stays uniformly bounded in n (§3). This means that in the exponentially long composition defining the renormalization,

$$hQ \cdots hQ \cdots hQhQ,$$

the h factors become linear on these tiny intervals and the Q factors have a bounded effect there. The inverse is basically a long composition of square roots—which yields Theorem 3. Now the real analysis is over and we must begin to work in the complex plane. We analyze compositions of square roots and schlicht mappings in §§5–7.

Theorem 3 provides preliminary information about holomorphic dynamics. (See Figure 3.) Notice the inverse branches of $f = hQ$ are of the form $S_\pm \cdot h^{-1}$, where $S_\pm$ are branches of $z \to \sqrt{z}$ composed with appropriate linear maps. These two branches have interior disjoint images. They fit along the boundary so we can form a forward mapping with domain of

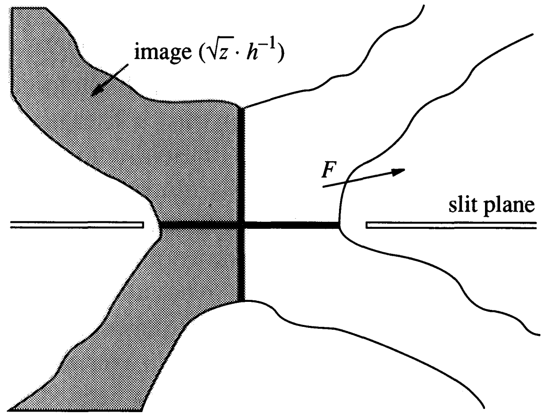


FIGURE 3

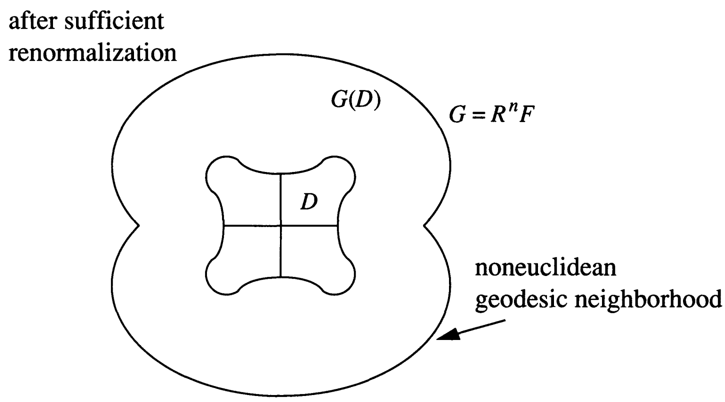


FIGURE 4

definition the union of these two images. We obtain a holomorphic mapping F extending f defined on a four-fold symmetric simply connected domain containing a definite neighborhood of the dynamic interval and mapping onto (the domain of h^{-1}) $= \mathbb{C} - \{x$ real not in $I(\lambda)\}$.

In §8, using §§3–7, we show how to cut down such a mapping after enough bounded time renormalization so it has the form of Figure 4.

THEOREM 4. *Assume bounded combinatorics* ($\leq T$) *and* F *belongs to the Epstein class* $E(\lambda)$. *After* $n \geq n(T)$ *renormalizations the inverse branches of* $G = R^n F$ *map the geodesic neighborhoods of Figure* 3 *well inside and the annulus* $G(D) - D$ *has a conformal modulus* $\geq m(T)$.

NOTE. The formulation of Theorem 4 and one of the key steps in its proof was motivated by papers of Epstein [**E1**, **E2**] and Epstein and Lascoux [**EL**]. A new point in the proof is a systematic use of information of Theorem 3 moving carefully up through the renormalization hierarchy.

The map depicted in Figure 4 is one of the *complex quadratic-like* *mappings* of Douady and Hubbard **[DH1]**. In the *conjugacy pullback argument* we make use of their insight that the annulus $G(D) - D$ with the ∂-relation given by G is a *fundamental domain* for the holomorphic dynamics of G. We go one step further (§9) and construct a Riemann surface lamination of orbits which in effect removes the branching locus singularity in the Douady-Hubbard fundamental domain modulo side identifications. In §11 one constructs a quasiconformal conjugacy whose distortion only depends on the universal bound on the conformal modulus of Theorem 4. The argument is reminiscent of the one in Michael Shub's thesis about expanding mappings. There are new twists—complex analyticity and quasiconformality replace the expanding property, branched coverings replace coverings so Thurston's insight about the role of the forward critical orbit is needed, and there is McMullen's remedy of the shrinking domain $D \supset G^{-1}D \supset G^{-2}D \supset \cdots$ using the Douady-Hubbard insight mentioned above.

Given quasiconformal conjugacies we can describe all the dynamical systems of one topological form by a Teichmüller space of conformal structures. The relevant Riemann surface lamination is studied in the appendix. Let $Q(R, T)$ be the complex quadratic-like mappings which are symmetric about a real axis and are infinitely renormalizable of bounded type $\leq T$. Let d be the Teichmüller distance (appendix) on the manifolds of quasiconformally conjugate systems. Then we can use the *almost geodesic lemma* (appendix) to prove Theorem 5 (§13).

THEOREM 5. *There is $\lambda(T) < 1$ so that for any two points x, y in $Q(R, T)$ there is a power of R that reduces d by a factor of λ,*

$$d(R^n(x), R^n(y)) \leq \lambda d(x, y).$$

The power depends only on the moduli of representatives of x and y (§10).

The renormalization R is defined canonically (§12) on $Q(R, T)$ respecting the operation R considered previously. It is clearly distance nonincreasing, $d(Rx, Ry) \leq d(x, y)$. More importantly the new R is defined on the level of germs of invariant conformal structures near the Julia set so *Teichmüller theory* applies.

With Theorems 3, 4, and 5 in place, the proof of Theorem 2 is a limit argument using the idea that f_0 is deeply embedded inside f_n for n large. In the type $(\tau, \tau, \tau, \dots)$ case Curt McMullen, motivated by the rigidity theory of Kleinian groups, has a different proof of *Theorem* 4 *implies Theorem* 2 using a geometric limit of this embedding idea (see **[Mc]**). Theorem $2'$ is really what the proof of Theorem 2 yields.

Very recently Edson de Faria **[deF]** found the replacement of complex quadratic-like mappings and the pullback construction in the context of critical circle mappings. Thus, one can expect versions of Theorems 2, 3, 4, and 5 in that context as well. Theorem 1 is known in that context by combining Swiatek's original argument **[Sw]** with the techniques of §3. Unbounded type can also be treated better for circle mappings because of work of Yoccoz **[Y1]** and Lanford **[L2]**.

Returning to folding mappings let us compare the results here with previous ones. First there is Lanford's computer assisted proof of the Feigenbaum calculations **[L1]**. It shows there is an analytic fixed point g of R_τ and the linearized version of R_τ, acting on a space of analytic functions at g, is hyperbolic with one eigenvalue $4.6692\ldots$ outside the unit circle and the rest inside. The small spectrum increases out to the unit circle as the regularity is decreased. According to Lanford on a sufficiently small real analytic neighborhood of g one has the expected hyperbolic picture for R_τ with stable manifold all the folding mappings in the neighborhood of type $(\tau, \tau, \ldots)$.

By the above we have a global version of Lanford's results. This fixed point g, or that of Epstein **[E1]** or of Campanino-Epstein-Ruelle **[CER]**, etc. must agree with $F(\ldots, \tau, \tau, \tau, \ldots)$ by the corollary to Theorem 2. At $F(\ldots, \tau, \tau, \tau, \ldots)$ we have the "stable manifold" $W^{\mathrm{s}}(\tau\tau\tau, \ldots)$ consisting of all smooth quadratic-like mappings of type $(\tau\tau\tau, \ldots)$. Its intersection with Lanford's real analytic neighborhood of g is Lanford's stable manifold. At $F(\ldots, \tau, \tau, \tau, \ldots)$ we have the "unstable manifold" $W^{\mathrm{u}}(\ldots, \tau, \tau, \tau; c)$. By the topological picture (Figure 2) this curve of canonical mappings constructed synthetically must extend Lanford's unstable manifold defined near g by the computer assisted hyperbolicity results.

The critical smoothness in Lanford's linear problem is certainly 1 + Hausdorff dimension of the critical orbit Cantor set—although only something weaker may be rigorously proven **[L3]**. By the main theorem this dimension is a universal value. In the case of $(\tau, \tau, \tau, \ldots)$ it is computed **[CCR]** to be $0.538045143580549911671415567\ldots$, but for the other types $(\sigma, \sigma, \sigma, \ldots)$ these dimensions vary in $(0, 1)$. So this fits with our smoothness class 1 + Zygmund which is contained in every $C^{1+\alpha}$ for $\alpha < 1$ but is bigger than first derivatives Lipschitz.

Theorem 1 has a precursor in Guckenheimer **[G1]** in the important special case of negative schwarzian f of period doubling combinatorics (see also **[VSK]**). In the circle mapping case the first part was obtained by Swiatek **[Sw]** and Herman **[H]**. There have been important generalizations of **[G1]** in Guckenheimer and Johnson **[GJ]** and of our method in Martens **[M]**. Other important papers involving such real bounds are Lyubich **[Ly2]**, de Melo and van Strien **[MV2]**, Blohk and Lyubich **[BL1, BL2]**, and Jakobson and Swiatek **[JS1]**. Many of these results and an outline of our proof can be found in **[MV]**.

I am grateful for the steadfast interest and guidance provided by my colleagues in Europe—Collet, Cvitanovic, Douady, Eckmann, Epstein, Herman, Lanford, Rand, van Strien, and Yoccoz—and in America—de Melo, Feigenbaum, Guckenheimer, Hubbard, McMullen, Swiatek, Tangerman, and Tresser. I would also like to express appreciation to the younger mathematicians Jiang, de Faria, and Paluba who listened to and critiqued arguments for untold hours over the last few years. Finally, I thank Wellington de Melo who has studied the proof relentlessly and provided crucial assists at various difficult points.

CONTENTS

1. Poincaré length distortion and smoothness class one plus Zygmund. In this section (in two stages) we show that the distortion of the cross ratio for standard 4-triples being O(scale of 4-triple) is equivalent to smoothness one plus Zygmund. Also, the smoothness implies control on cross ratio distortions for sufficiently many nonstandard 4-triples to yield the dynamical Koebe distortion argument of §2.

We want to study the smoothness required for a diffeomorphism h to only distort cross ratios of small standard 4-tuples by an amount commensurable to the size of the 4-tuple.

One cross ratio $[a, b, c, d]$ can be computed by

$$-\log[a, b, c, d] = \iint_S \frac{dx\,dy}{(x-y)^2}, \qquad a < b < c < d,$$

where S is $\{(x, y) | a \leq x \leq b,\ c \leq y \leq d\}$.

Thus the distortion by h, given by

$$\log \frac{[ha, hb, hc, hd]}{[a, b, c, d]},$$

equals $\int_S \mu - (h \times h)^* \mu$, where μ is the measure, $dx\,dy/(x-y)^2$.

Calculating the integrand we get

$$\left(\frac{1}{(x-y)^2} - \frac{h'xh'y}{(hx-hy)^2}\right) \quad \text{or} \quad \frac{1}{(x-y)^2}\left[1 - \frac{h'xh'y}{[h']_{xy}^2}\right],$$

where $[h']_{xy} = \text{average}_{[x,y]}h'$ is the average of h' over the interval $[x, y]$.

Because we are assuming $b-a=c-b=d-c$, for every point (x, y) in the square S the factor $1/(x-y)^2$ is commensurable to $1/\text{area}\,S$. Thus, a small bound ε on $\log(h'xh'y/[h']_{xy}^2)$ yields the bound ε on the distortion of the cross ratio, $\log[ha, hb, hc, hd]/[a, b, c, d]$.

Let us say h satisfies the *local Koebe condition* if for $|x-y|$ sufficiently small one of the following equivalent conditions holds:

(1) $[1-(h'xh'y/[h']_{xy}^2)] = O(|x-y|)$,

(2) $\log(h'xh'y/[h']_{xy}^2) = O(|x-y|)$.

PROPOSITION. *If h satisfies the local Koebe condition, then the h distortion of cross ratios of small standard 4-tuples is commensurable to the size of the 4-tuple.*

PROOF. The proposition follows from the above calculations. Q.E.D.

Calculating the log in (2) we get

$$\log h'x + \log h'y - 2\log[h']_{xy}.$$

Let us replace the last term with the average taken after the log to obtain (a) $(\log h'x + \log h'y - 2[\log h']_{xy})$ with an error of twice (b) $(\log \text{average}_{[x,y]}(h') - \text{average}_{[x,y]}(\log h'))$.

NOTE. If both (a) and (b) are $O(|x-y|)$, then (1) and (2) hold.

Expression (a) suggests the Zygmund condition on continuous functions:

$$Z\colon \varphi(x) + \varphi(y) - 2\varphi\left(\frac{x+y}{2}\right) = O(|x-y|).$$

PROPOSITION. *If φ satisfies Z on an interval J, then the average of φ over J is the value of φ at the midpoint with an error $O(\text{length } J)$.*

PROOF. Think of the uniform measure on J as two dirac masses moving out uniformly from the center. Use the Z condition to replace the average of φ at the moving points by the value at the center. Q.E.D.

COROLLARY. *If $\log h'$ is Zygmund, then expression* (a) *is* $O(|x-y|)$.

PROOF. Use the proposition, then the definition of Z again.

There is a converse to the corollary. Say φ satisfies the *average property* if $\text{average}_{[x,y]}\varphi = \frac{1}{2}(\varphi(x)+\varphi(y)) + O(|x-y|)$.

PROPOSITION. *If φ satisfies the average property for all intervals $J \subset I$, then φ satisfies the Zygmund property for all pairs x, y in I.*

PROOF. Apply the average property to $[x, (x+y)/2]$, $[(x+y)/2, y]$, and $[x, y]$, and combine averages of averages to get the Zygmund property for x, y.

COROLLARY. *The Zygmund property is equivalent to the average property.*

PROOF. The corollary follows from the propositions above.

CONCLUSION A. Expression (a) is $O(|x-y|)$ iff $\log h'$ is Zygmund.

Now we consider when expression (b) is $O(|x-y|)$. We are concerned with small intervals J and we assume h' is continuous. Then h' varies only a little from one of its values $h'(x_0) = a$. Expression (b) is unchanged if we multiply $h'x$ by $1/a$. Write $(1/a)h'$ on J as $1+\varepsilon$, where ε is a small function. Expand the two terms of (b) as:

$$\begin{aligned}
&\log \frac{1}{|J|}\int_J (1+\varepsilon) - \frac{1}{|J|}\int_J \log(1+\varepsilon) \\
&= \left(\frac{1}{|J|}\int_J \varepsilon - \frac{1}{2}\left(\frac{1}{|J|}\int_J \varepsilon\right)^2 \cdots\right) - \left(\frac{1}{|J|}\int_J \varepsilon - \frac{\varepsilon^2}{2}\cdots\right) \\
&= -\frac{1}{2}\left(\frac{1}{|J|}\int_J \varepsilon\right)^2 + \frac{1}{|J|}\int_J \varepsilon^2/2 \cdots .
\end{aligned}$$

Here the first term could be zero so there would be no cancellation. Thus, we estimate each brutally with absolute values. Assume ε is Hölder of order $\frac{1}{2}$ on J, $|\varepsilon(x) - \varepsilon(y)|^2 \le C_J|x-y|$. Since ε is zero at x_0, we get the estimate $C_J \cdot \text{length}\, J$ for the sum of the absolute values. Also, if $C_J \cdot \text{length}\, J$ is sufficiently small, then the higher order terms can be ignored.

CONCLUSION B. Expression (b) is $O(|x-y|)$ if h' is Hölder of order $\frac{1}{2}$. The coefficient for $|x-y| < \varepsilon$ is estimated by the *normalized $\frac{1}{2}$-Hölder norm*: take the sup over all intervals J of length $\le \varepsilon$ of C_J above, where $1+\varepsilon = h'(x)/h'(x_0)$ for convenient x_0 in J and we assume $C_J|\text{length}\, J|$ is sufficiently small.

Let us note that Zygmund functions are α-Hölder for all $\alpha < 1$. However, the α-Hölder constants are not determined by the Zygmund norm. Let us also note the normalized $\frac{1}{2}$-Hölder norm of h' can be estimated by the square of the usual $\frac{1}{2}$-Hölder norm of $\log h'$ – the best C such that

$$|\log h'x - \log h'y|^2 \le C|x-y|.$$

Now we can summarize the above by the following theorem:

THEOREM. (a) *If* $\log h'$ *is Zygmund, then h satisfies the* local Koebe distortion *condition. The coefficient is controlled by the Zygmund norm of* $\log h'$ *and the $\frac{1}{2}$-Hölder norm of* $\log h'$. *Conversely,*

(b) *if* $\log h'$ *is* $\frac{1}{2}$*-Hölder, then the local Koebe condition for* h *implies* $\log h'$ *is Zygmund.* (*See Remark below for a stronger statement.*)

PROOF. The above discussion has been a proof of (a). For part (b) recall from above that the local Koebe inequality implies that expression (a) plus expression (b) is $O(|x-y|)$. The $\frac{1}{2}$-Hölder norm implies expression (b) is $O(|x-y|)$. Thus, expression (a) is $O(|x-y|)$. But this implies $\log h'$ is Zygmund by the third proposition above. Q.E.D.

Other results about cross ratio distortion can be found in **[MV]**.

PROBLEM. Derive necessary and sufficient conditions for the integral distortion to be commensurable to the linear scale. (In the above discussion we have estimated the integral by the integrand.)

REMARK (added December 1990). Actually we can solve this problem. By a standard 4-triple (a, b, c, d) we mean one where $(b-a)=(c-b)=(d-c)$. The solution of the problem is to show *a homeomorphism* h *which distorts a standard tiny* 4*-triples' cross ratio by* $O(scale)$ *is a diffeomorphism with* $\log h'$ *Zygmund, and conversely.* We sketch this.

The same method shows $O(\text{scale})^{\alpha}$ *cross ratio distortion is equivalent to* $\log h'$ *is* C^{α}, $0<\alpha<1$, *and* $C^{1,\alpha-1}$, $1<\alpha\le 2$, *while distortion* $o(h^2)$ *is equivalent to* h *being Moebius.*

The proof consists of studying for a fine grid of intervals I_β the approximate derivatives $d_\beta = |hI_\beta|/|I_\beta|$, the ratio distortions $r_\beta = |hI_{\beta'}|/|hI_\beta|$ for consecutive intervals, and the cross ratio distortions c_β = change in $\ln(1+(|I_{\beta'}|(|I_\beta|+|I_{\beta'}|+|I_{\beta''}|))/|I_\beta|\,|I_{\beta''}|)$, where I_β, $I_{\beta'}$, $I_{\beta''}$ are consecutive intervals. If $l_\beta = \frac{1}{2}\log d_\beta$ and $\varepsilon_\beta = \frac{1}{2}\log r_\beta$, then

(1) $\varepsilon_\beta = l_{\beta'} - l_\beta$ exactly,

(2) $c_\beta = \varepsilon_{\beta'} - \varepsilon_\beta$ modulo higher order terms in $\varepsilon_{\beta'}$.

One may also compare maximum ratio distortions $\varepsilon(t)$ at two adjacent scales and find

(3) $\varepsilon(2t) \ge 2\varepsilon(t) + c(t) +$ higher order terms in $\varepsilon(t)$, where $c(t)$ is the maximum cross ratio distortion at scale t.

To use these relationships there are two important preparation lemmas:

LEMMA 1. *If* $c(t)$ *is bounded on some open interval, then* $\varepsilon(t)$ *is bounded on every closed subinterval.*

LEMMA 2. *If* $c(t)$ *tends to zero on some open interval, then* $\varepsilon(t)$ *tends to zero in every closed subinterval.*

Lemma 1 follows from the *four intervals remark*: if for four consecutive equal intervals the middle two are mapped to very disparate intervals, then the Poincaré length $\ln(1+\frac{MT}{LR})$ of one interval M in three others $T = L+M+R$ is greatly increased by the map (see §2).

Lemma 2 uses the exact relationship between c_β and ε_β to say that if $\varepsilon_\beta > 0$, then $\varepsilon_{\beta'} > \varepsilon_\beta$ mod terms the size of c_β. Then at twice the scale one gets at least twice the ratio distortion. Using this amalgamation many times contradicts the boundedness of ratio distortion unless $\varepsilon_\beta \to 0$ with the scale.

Now if $c(t) = O(t^\alpha)$, $0 < \alpha < 1$, one uses (3) to get $\varepsilon(t) = O(t^\alpha)$, which is equivalent to h being a $C^{1+\alpha}$ diffeomorphism.

If $c(t) = O(t)$ we use (1) and (2) to get

(4) $l_{\beta''} + l_\beta - 2l_{\beta'} = O(t)$ plus terms $O(t^{2\alpha})$, $\alpha < 1$. Since $\varphi = \log h'$ is α-Hölder for all $\alpha < 1$, we have $|\frac{1}{|I|}\int \log h' - \log \frac{1}{|I|}\int h'| = O(t^{2\alpha})$ for all $\alpha < 1$.

Thus for $\log h'$ (4) implies that the average over an interval I is the average over the middle $\frac{1}{3}$ with error $O(|I|)$. Iterating this yields that the average of φ over I equals the value of φ at the midpoint with error $O(|I|)$. Thus, $\varphi = \log h'$ satisfies the Zygmund condition,

(5) $\varphi(x+t) + \varphi(x-t) - 2\varphi(x) = O(t)$.

Continuing, if $c(t) = O(t^\alpha)$ for $1 < \alpha < 2$, one gets versions of (4) and then (5) with $O(t)$ replaced by $O(t^\alpha)$. Dividing by t and looking at a geometric series over the scales yields $\varphi' x$ is $C^{\alpha-1}$. Now if $\alpha = 2$, we can have (4) with error $O(t^2)$ and then (5) with error $O(t^2)$, which by the same argument gives φ' is Lipschitz.

If $c(t)$ is $O(t^2)$, one can try to define the Schwarzian as $\lim c(t)/t^2$. Thus, $c(t) = o(t^2)$ means h is Moebius.

Remark. These calculus results can be used to prove familiar results about circle diffeomorphisms f for optimal levels of smoothness in the above scale. The following is a scorecard:

	$C^{1+\alpha}$, $0<\alpha<1$	$C^{1+\text{Zygmund}}$	$C^{1+\text{zygmund}}$
cross ratio distortion	$O(t^\alpha)$	$O(t)$	$o(t)$
Denjoy's theorem	no	yes	yes
ergodicity of f	no	yes	yes
renormalization is bounded	no	yes	yes
renormalization limits are rotations	no	no	yes
M. Hermann ratio rigidity for bounded type	no	no	yes

See forthcoming notes from CUNY.

2. The Koebe distortion argument of Denjoy, de Melo-Van Strien, Swiatek, Yoccoz, et al. and Zygmund smoothness. Consider a composition g of many diffeomorphisms f_i between tiny intervals J_i all lying disjointly in some big interval I, $f_i: J_i \to J_{i+1}$.

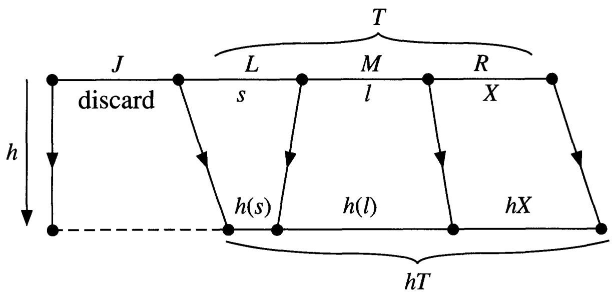


FIGURE 1

The classical Denjoy argument estimates $\log|g'x/g'y|$, $x, y \in$ domain g, in terms of the $\sum_i$ total variation $|\log f_i'|$. This will be finite, say, if $f_i = f/I_i$ and $\log f'$ has bounded variation on I. The proof is by the chain rule.

The new argument, called the Koebe principle for one-dimensional real dynamics, treats the case when the factors can be divided into two groups so that relative to some coordinate system on I

(i) for one group a Denjoy type argument can be used at least to study cross ratios.

(ii) the factors in the other group decrease Poincaré length (a type of cross ratio) (because of a positive Schwarzian condition) even though $\log f_i'$ can have unbounded variation.

Here if L, M, R is a partition of an interval T into three consecutive subintervals (the left, the middle, and the right) the *Poincaré length of M in T* is $\log(1+\frac{MT}{LR})$. It is the length of M in the Riemannian metric on $T=[a, b]$ corresponding to the form $|dx|/(x-a)+|dx|/(b-x)$.

The additive change of Poincaré length (P-length) along a composition is additive over the factors. *In a decomposition such as* (i), (ii) *above, the increase in P-length is controlled by the factors of type* (i) *because there is a decrease for the factors of type* (ii). *This is the first idea* (cf. Swiatek [**Sw**]).

The *second idea is the four intervals argument.* Let J, L, M, R be contiguous equal length intervals and let h be a homeomorphism of the union into the real line so that one of hL and hM is *much smaller* than the other. Discard from the original four intervals the outer interval next to the one of L or M made smaller, called s. Let T denote the union of the remaining three L, M, X and let $l \subset T$ be the one of L or M made larger. The P-length of $l \subset T$ is log 4. The P-length of $h(l) \subset hT$ is very large because $h(l)$ is much larger than $h(s)$ and $h(T)$ is of course greater than hX. Thus, one has the analogue of complex Koebe distortion:

REAL KOEBE DISTORTION. *If a homeomorphism $h: I \to$ reals does not increase unit P-lengths too much the quasisymmetric distortion of interior symmetric triples is controlled.*

More precisely, if $x, y \in I$ satisfy that $|x-y|$ is as small as the distance to ∂I and $z=(x+y)/2$, then $1/M \le (h(x)-h(z))/(hy-hz) \le M$, where

M can be calculated from the bound B on the additive increase of Poincaré length of unit Poincaré length subintervals $J \subset T$ where $T \subset I$, i.e., the B defined by

$$(\text{P-length of } hJ \subset hT - \text{P-length } J \subset T)_+ \leq B$$

for all $J \subset T$ so that P-length $J \subset T = 1$.

REMARK. The point here as in Koebe distortion for schlicht mappings is we go from one analytic condition (in that case holomorphic; in this case positive Schwarzian or controlled P-length increase) to interior control on the nonlinearity.

We describe the dynamic Koebe distortion principle for a rather general class of dynamic systems. Let M be a compact one-manifold provided with a differentiable structure, where overlap homeomorphisms $h_{\alpha\beta}$ are continuously differentiable and the $\log h'_{\alpha\beta}$ have bounded Zygmund norm (see §1).

Suppose $f: M \to M$ is a smooth mapping with finitely many critical points where $f' = 0$. At a nonsingular point assume $\log f'$ is Zygmund. At a singular point c_i suppose there are coordinate systems in the $(1+\text{Zygmund})$ structure so that f takes the form $x \to |x|^{r_i} + v_i$ or $x \to (\operatorname{sign} x)(|x|^{r_i}) + v_i$, where $r_i > 1$.

Assume we have a long composition g of diffeomorphisms $f_i: J_i \to J_{i+1}$ where the J_i are disjoint in M and $f_i^{-1} = f$ restricted to J_{i+1}. Quasisymmetric distortion is defined informally above and formally in §3.

THEOREM. *For the composition g the increase in Poincaré length and therefore the interior quasisymmetric distortion of g in domain g is controlled by constants of the coordinate systems and local models of f, independent of the length of the composition g.*

PROOF. We first need a lemma.

LEMMA. *If h is a diffeomorphism of the unit interval I, $\log h'$ is Zygmund, $T \subset I$ is a tiny interval, and $J \subset T$ has unit Poincaré length, then the Poincaré length of $hJ \subset hT$ is $1 + O(\text{length}\, T)$. The coefficient is controlled by the Zygmund norm of $\log h'$ and the $\frac{1}{2}$-Hölder norm of $\log h'$ squared.*

PROOF OF THE LEMMA. We have proved this in §1 when J sits in the middle of T. In general J may be tiny and near one end of T. We have to calculate the integral of §1 over the rectangle R of Figure 2. Control on the integral yields that the control on P-length changes for 4-triples of P-length ~ 1.

Using the local Koebe condition, and the fact that for a point in R the distance to the diagonal and the vertical distance to the diagonal are equivalent, the integral takes the form

$$a \cdot \int_a^b \frac{1}{t^2} O(t)\, dt,$$

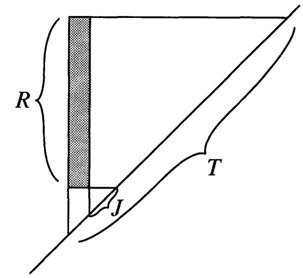


FIGURE 2

where $a \sim \text{length}\,J \sim \text{distance}(J, \partial T)$ and $b \sim \text{length}\,T$. This yields $a \log b/a$ which has order b when a and b are commensurable. This is the case already discussed. Otherwise, if $a \ll b$, $a \log b/a$ is much smaller than b. This proves the lemma.

PROOF OF THEOREM. (i) As we go along the composition a Poincaré length is decreased if we are entirely within one of the coordinate systems for the singular point models because f^{-1} has positive schwarzian there and maps of positive schwarzian decrease Poincaré length (de Melo and Van Strien **[MV1]**, **[MV2]**).

(ii) There are finitely many possible transitional cases for long intervals which do not fit inside one model or the other. We will not discuss these further. They are finite.

(iii) Finally we have the factors where the lemma applies. We view the lemma as saying intervals $J \subset T$ of any P-length ≥ 1 cannot increase by more than the multiplicative factor $1 + O(\text{length}\,T)$. By disjointness of the orbit of T this effect is controlled by the total length of M. Q.E.D.

For an alternative exposition of the theorem see **[MV]**.

3. The a priori real bounds (proof of Theorem 1).. Let $f\colon I \to I$, $f(a) = f(b) = a$, $I = [a, b]$, be a smooth quadratic-like mapping, i.e., $f = Qh$, where $Q\colon I \to I$ is a quadratic polynomial and $h\colon I \to I$ is a diffeomorphism with $\log h'$ Zygmund. Recall we bound h in terms of $\varphi = \log h'$ by the best B so that for all x, y in I,

$$\left|\frac{\varphi(x) + \varphi(y)}{2} - \varphi\left(\frac{x+y}{2}\right)\right| \leq B|x-y|, \qquad |\varphi(x) - \varphi(y)|^2 \leq B|x-y|.$$

We say that f is bounded by B.

THEOREM 1 ("beau" property of renormalization). *Any sequence of renormalizations of f is bounded in terms of B and after a number of renormalizations depending on B the bound is universal and independent of B.*

PROOF. We will continually meet bounds with the properties of the theorem. We say such a quantity is "beau" (bounded and eventually universal). We will number some useful statements developed along the way. Let

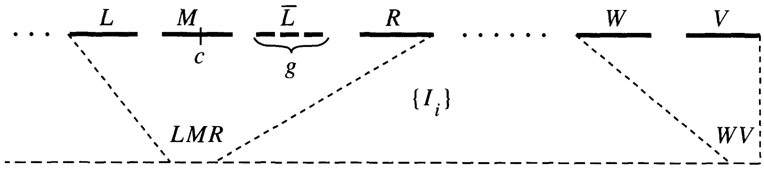


FIGURE 1

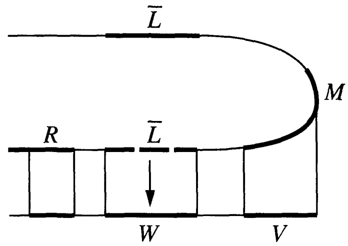


FIGURE 2

$\{1, 2, 3, \dots\}$ denote the forward orbit of the critical point c and let I_j or $I(j)$ for $j = 1, 2, \dots, q_n$ denote the intervals bounded by $\{j, j + q_n\}$, where the nth renormalization of f is defined by $R^n f = f^{q_n}/I'$ and $I' \supset I(q_n)$. In Figure 1 M is $I(q_n)$, the interval containing the critical point, L is its neighbor among the I_j mapping to the neighbor W of $V = I_1$, the interval containing the critical value, and R is the other neighbor among the I_j of M. $\bar{L}$ is the mirror image of L and it lies in the gap g between M and R (see Figure 2).

To derive the "beau" property of the bounds we first go through all the steps to get some bound, then we go back through the steps again to achieve the eventually universal property. Let (I, J) denote the smallest interval containing I and J.

(1) *There is $\lambda > 1$ independent of n so that diameter$(L, M) \geq \lambda$ length M.*

PROOF OF (1). Suppose I_s among the I_j is smallest, and $s > 2$. By the next item (2) $f^{s-1}: I_1 \to I_s$ has an inverse branch defined on I_s and its immediate neighbors, which are longer than I_s. Using Koebe (§2), diameter$(W, V) \geq \lambda'$ length V.

Now we use the fact that h^{-1} controlled by B and Q^{-1} has bounded quasisymmetric distortion to do the one more preimage required and to treat the cases $s = 1$ and $s = 2$.

DEFINITION. The *quasisymmetric distortion* $M(q)$ of a homeomorphism q of an interval I is the sup of $\log|(q(x) - q(y))/(q(y) - q(z))|$ over all symmetric triples x, y, z in I.

(2) *$f^j: I_1 \to I_{j+1}$ has a continuous inverse branch defined on the span of I_{j+1} and its immediate neighbors among the I_α.*

FIGURE 3

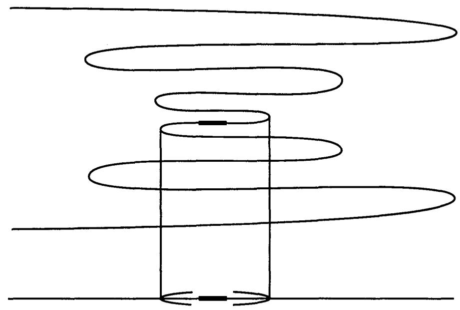


FIGURE 4

PROOF OF (2). Each endpoint of an I_j is the image of the fold and these are oriented as in Figure 3. This is true for I_n because $g = f^{q_n}/I_n$ is a unimodal map and its endpoints are the critical value of g and its image. It is then seen for the other intervals by applying the dynamics.

Now any inverse branch defined near x for the kth power of a folding map has a maximal interval of definition which is bounded by either endpoints of the original interval I or by the forward images of the critical point whose folds point *away* from x. See Figure 4.

For f_j these limiting points cannot be the endpoints of I_{j+1} which are $\{j+1, j+1+q_n\}$ because j is smaller than each. Thus, these can only be the outer endpoints of the neighbors or further away. Q.E.D.

(3) REMARK. It seems we cannot proceed without the combinatorial fact (2) which was the breakthrough point for Theorem 1.

(4) *Any composition of inverse branches of length l starting at I_j for $l < j$ has quasisymmetric distortion bounded on I_j plus a definite proportional neighborhood.*

PROOF OF (4). Any f^{-j} for $j < q_n$ is defined on span (LMR) by (2) so by (1) and Koebe all these have bounded quasisymmetric distortion on a definite neighborhood of M. Call this extra room the "Koebe space." Now we can move the Koebe space around M around to each of the intervals and apply Koebe again to obtain the proof of (4).

(5) Now we would like to repeat the above argumentation using the bigger (official) renormalization intervals I'_α, namely the appropriate subcollection of the inverse orbit of $I' \subset I$, where $g = f^{q_n}/I'$ is a smooth quadratic-like mapping, in the above sense, $g(\partial I') \subset \partial I'$. These intervals are interior disjoint and bounded by the points of a periodic orbit and some of their preimages. A modification of the above is required. Point (2) fails literally. However it only fails finitely. Namely, in the inverse of $f^j : I'_1 \to I'_{j+1}$ there

$$\ldots \quad L' \quad M' \quad \overline{L'} \quad R' \quad \ldots\ldots \quad W' \quad V'$$

FIGURE 5

may be one or two Q^{-1} factors with poles in images of the neighbors of I'_{j+1}. Thus, we may factor f^{-j} on the span of I'_{j+1} and its immediate neighbors into at most three compositions where Koebe (§2) applies, interposing at most two quasisymmetric maps given by the Q^{-1} factors. There is one dangerous possibility that must be excluded. If the first Q^{-1} factor cuts away most of the Koebe space it follows that $\operatorname{diam}(L', M')$ is very large compared to $\operatorname{diam} M'$, where we use the prime analogue of the notation of Figure 1 (see Figure 5). If so we pick up the previous argumentation at point (4). Then if at some point Koebe space is cut away as we pull back the triple of intervals $(L', M', \overline{L}')$, then for the composition up to this point we must have $L' \to \cdots \to V'$ and $M' \to \cdots \to W' \to L'$. The second composition is qs by Koebe. Thus, the relevant critical point is quasicentered in L' because the critical point of M' is centered in M'. It follows by Koebe the first composition is qs on part of L' between the critical point and M'. Then the critical point of V' is not too close to W' relative to $\operatorname{diam}(W')$. Thus, the loss of Koebe space is controlled at this one dangerous moment, which is the only point different from the previous argumentation.

This proves we have statement (4) with I'_α replacing I_α.

(6) Note we have shown the analogue of statement (1) with the corresponding prime notation: *There is a $\lambda' > 1$ independent of n so that diameter $(L', M') \geq \lambda'$ length M'.*

(7) *The total length of the intervals I'_α decreases by a definite factor each time we renormalize.*

PROOF OF (7). Consider a sequence of renormalizations of combinatorics $\sigma_1, \sigma_2, \ldots, \sigma_k$ of degrees of $n_1, n_2, \ldots, n_k$. Apply point (6) to the composed renormalization to find a gap next to the critical point interval of size comparable to it. By point (5) we can move this gap around to be adjacent to each of $n_1, n_2, \ldots, n_k$ intervals.

Now think of this $\sigma_1, \sigma_2, \ldots, \sigma_k$ renormalization as a σ_k renormalization of the $\sigma_1, \ldots, \sigma_{k-1}$ renormalization. We see these moved around gaps among the new gaps of the σ_k-renormalization. This shows the length decreases by a definite factor because of our constructed gaps in each σ_k packet.

(8) *Each of the bounds is "beau."*

PROOF OF (8). The total length of the intervals goes to zero at an exponential rate whose constants depend on B. But then the accumulated effect of h on the above bounds tends to zero exponentially fast at a rate depending on B. All the other considerations were independent of f and B. Thus, any bound derived as above is "beau."

(9) *In a linear coordinate system where $x = 0$ is the critical value define*

the nonlinearity of an interval J *to be* $\int_J |dx/x| = n(J)$. *Then* $\sum' n(I'_\alpha)^2 = O(1)$, *where we sum over all the intervals except the critical value interval—which has infinite nonlinearity.*

PROOF OF (9). If we move around through the entire circuit we can study how the Poincaré lengths of I'_{q_n}, in a neighborhood where the inverse branches are injective, are altered. We can choose this neighborhood to be definite since each $I_j \subset I'_j$ contains a point roughly centered (because this is true for $j = q_n$, then use (5)) and we have (2). The effect on Poincaré length is bounded by (5) on the one hand and can be calculated in terms of the sum and a bounded effect due to h. By the same reasoning as in (8) the bound is "beau."

(10) Now look at the entire renormalization down to some level. We see a long composition of h's and Q's making up the diffeomorphism h_n in the decomposition $R^n f = Qh_n$. All the partial compositions are uniformly quasisymmetric by (5). Thus, they are uniformly Hölder of some exponent. A Q^{-1} factor on I_α has B-norm $\sim (nI_\alpha)^2$ and the total B-norm is bounded by (9).

Consider each factor of the composition as acting on a definition neighborhood of its dynamical interval among the I'_α. Rescale each of these larger intervals to be a standard interval and consider an exponentially small subinterval J. For each partial composition the image of J is also exponentially small by the the uniform Hölder property coming from the uniform quasisymmetric bound.

Applying the Zygmund control on h and (9) we see the distortion of standard cross ratio on the scale of J is exponentially small. This implies (last remark §1) that on the rescaled dynamic interval inside the standard interval the quasisymmetric distortion is exponentially small. This calculus exercise shows such homeomorphisms are uniformly $C^{1+\alpha}$, for appropriate α (and conversely). We take from this that the first derivatives of all partial compositions of the rescaled dynamic intervals I'_α to itself are on the order of one. Repeating the cross ratio argument using this Lipschitz control yields $O(\text{scale})$ distortion of cross ratio of standard 4-tuples. Lipschitz plus $O(\text{scale})$ control implies B-bounded on a closed subinterval (§1). Q.E.D.

4. Renormalization limits and schlicht mappings—the Epstein class. Let us go one step beyond the "beau" property of the sequence of renormalizations of f, discussed in the previous section. We assume $f: I \to I$ is a smooth quadratic-like mapping bounded by B which is infinitely renormalizable with combinatorics $\sigma = (\sigma_1, \sigma_2, \dots)$. Let $f_1, f_2, \dots$ be the sequence of renormalizations of f and write $f_n = Q_n h_n$, where Q_n is the quadratic polynomial $Q_n: I \to I$ satisfying $Q_n(a) = Q_n(b) = a$ and $Q_n(c') = f_n(c)$, where c' is the critical point of Q_n and c is the critical point of f, $h_n: I \to I$ is a diffeomorphism, and $\partial I = \{a, b\}$.

THEOREM. *The family of renormalizations* $\{f_n = Q_n h_n\}$ *is precompact in the sense that the critical value of* Q_n *is bounded away from* a *and* $\{h_n\}$ *is precompact in the* $C^{1+\alpha}$ *topology on diffeomorphisms for any* $\alpha < 1$. *Any* C^0 *limit of the* f_n *is a folding mapping in the symmetric form* hQ *and* h^{-1} *has a complex analytic injective extension to* $\mathbb{C} - \{x$ *real but not in a universal neighborhood of* $I\}$. *Also,* $Q_n \to Q$ *and* $h_n \to h$ *in the* $C^{1+\alpha}$ *topology for any* $\alpha < 1$.

DEFINITION. If $J \supset I$ is the neighborhood of I where h^{-1} of the theorem is defined we say $f = hQ$ belongs to the Epstein class and we write $f \in E(J)$.

REMARK. We will see later the limit only depends in the bounded combinatorics case on the type $\sigma = (\sigma_1, \sigma_2, \dots)$ of f.

PROOF. In the long composition

$$R^n f = Qh \cdots QhQhQh/I_{q_n}$$

($2q_n$ factors) all the h factors are becoming linear exponentially fast in n in the sense that the sum of all their B-bounds is tending to zero exponentially fast (see (9), §3). Also all the partial compositions up to the last Q on the right are B-bounded by the work of the last section. In fact all this was seen to be true in a definitely larger neighborhood. Using the Lipschitz property of composition as a map from $C^{1+\alpha} \times C^{1+\alpha} \to C^{\alpha}$ we can remove the h factors one at a time. Thus, $R^n f$ can be written in the Epstein form with an exponentially small error in every C^{α}, $\alpha < 1$. Note the resultant composition of Q's being schlicht is controlled by its action on two interior points of the interval.

Then we use the corollary below and the boundedness of h_n to see that $Q'(a)$ is bounded from below. Thus, Q_n is bounded away from the zero quadratic polynomial. We can form limits and the above estimates prove what we want. Q.E.D.

Now we turn to the corollary which needs a lemma:

Let $f\colon I \to I$ be any folding mapping with $f(a) = f(b) = a$, $\partial I = \{a, b\}$.

LEMMA. *Either* (a) *there is no fixed point between the boundary fixed point and the critical value*; (b) *there is a smaller box* (*as in Figure* 1(c) *of the introduction understood for* $n = 1$); *or* (c) *no renormalization is possible, for* $n > 1$.

PROOF. If there is a fixed point p between a and c, consider the box on p. If it contains the graph we have a smaller box. Otherwise, the critical value in two iterates lands in the invariant interval $[(a, p)]$. In this second case there is no interval about the critical point whose images are interior disjoint and which returns to itself under some iterate.

COROLLARY. *For a minimal renormalization f which is further renormalizable, $f(x) > x$ for $a < x < c$ (if $f(c)$ is a maximum).*

PROOF. By the lemma $f(x) = x$ for some $\alpha < x < c$ implies that either there is a smaller box or no further renormalization is possible. The other possibility $f(x) < x$ for $a < x < c$ implies the critical value is forward asymptotic to a, so no further renormalization is possible.

(CONVERSE) REMARK. A box for f^n satisfying $f^n(x) > x$ for $a_n < x < c$ has interior disjoint images under the preimages of f following the critical orbit backwards.

PROOF. These preimage boxes define conjugate boxes. These can only overlap at the orientation preserving fixed point.

5. Composition of roots and the sector theorem. Let $[a, b] = I$ be an interval on the real axis. Let $S = S(a, b)$ be the set of injective holomorphic mappings (schlicht mappings) of $\mathbb{C}-\{x \text{ real not in } [a, b]\}$ which are homeomorphisms of $[a, b]$ to itself and preserve the two half planes. Then $S(a, b)$ contains left and right square roots, branches of $\sqrt{z}$ followed by linear transformations defined on $\mathbb{C}$ – slit, where the slit is a real ray complementary to $[a, b]$.

We consider a composition of elements from $S(a, b)$ of the form

$$A_n C_n \cdots A_2 C_2 A_1 C_1$$

satisfying:

(i) $A_1, A_2, \dots, A_n$ are left square roots and $C_1, C_2, \dots$ are general elements of $S(a, b)$.

(ii) A_1 has a singularity at a (i.e., the slit for A_i is $(-\infty, a)$), and $a_i =$ singularity of A_i moves away to the left exponentially fast, i.e., if $|a-b| = 1$, then $k \le |a_{i+1} - a|(|a_i - a|)^{-1} \le K$ for $i = 2, 3, \dots, 1 < k < K < \infty$.

(iii) If I_i denotes the maximal open interval on which C_i extends to a diffeomorphism into the reals, then $C_i I_i$ contains (a_i, a). Moreover, if $J_i' = J_i$ plus λ-proporational space on either side, where $J_i = C_i^{-1}(a_i, a)$, then $J_i' \subset I_i$.

SECTOR THEOREM. *There is a θ depending only on (k, K, λ) so that the image of the upper half plane by the composition $A_n C_n \cdots A_1 C_1$ is contained in the sector $0 \le \arg(z - a) \le \pi - \theta$ (see Figure 1 on next page).*

PROOF. (i) The regions of the upper half plane cut out by circles passing through a and b are Poincaré metric distance R neighborhoods of the geodesic (a, b) in $\mathbb{C} - \{x \text{ real but not in } (a, b)\}$. By Schwarz's lemma these are mapped into themselves by any element or any composition from S (Figure 2 on next page).

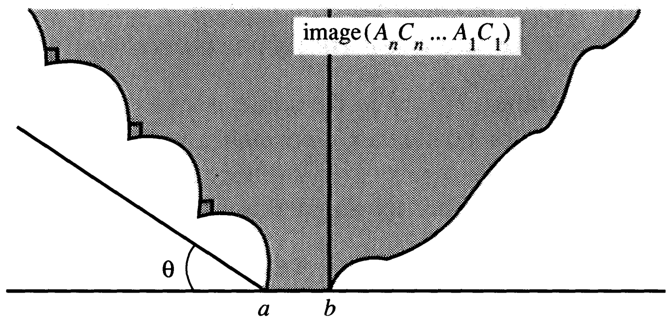


FIGURE 1

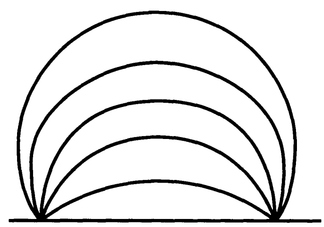


FIGURE 2

(ii) The composition C_i is injective on $\mathbb{C}-\{$ real x not in $I_i\}$ (assumption (ii)) and $J_i' \subset I_i$. So by Koebe distortion C_i yields a bounded from linear distortion mapping of a bounded shape neighborhood in the upper half plane U_i of J_i to a bounded shape neighborhood in the upper half plane V_i of (a_i, b) (Figure 3). Since C_i fixes the two points $\{a, b\}$ it only distorts the Euclidean metric on the neighborhood by a bounded factor.

Now we prove the sector theorem. The statement is obvious for $n = 1$, so assume $n \geq 2$.

Start with any point p_1 in the upper half plane and define p_2 such that $p_2 = A_1C_1p_1, \ldots, p_{i+1} = A_iC_ip_i$ for $i \geq 2$. First note that p_2 lies to the right or on a vertical line at a. There are two cases as i increases:

Case 1. p_i is far from (a, b) relative to the singularity a_i of A_i. Precisely p_i does not belong to U_i. Then p_{i+1} is not in $A_iC_iU_i = A_iV_i$. Now A_iV_i contains all the points of the strip $\{y > 0, x$ in $(A_ia_i, a)\}$ in a rectangle resting on (A_ia_i, a) with height a definite fraction of the base. Thus, the angle of p_i as viewed from a *is large* and *stays large* after application of A_iC_i. This is so because $A_iC_ip_i = p_{i+1}$ lies to the *right* of the vertical line at A_ia_i and *above* the rectangle at the bottom of the strip (Figure 4).

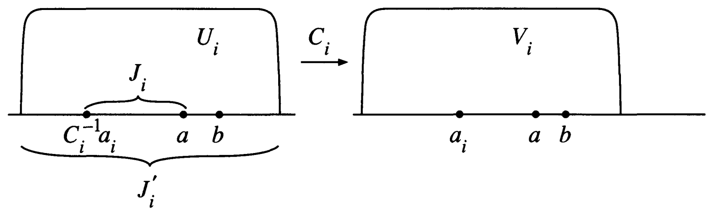


FIGURE 3

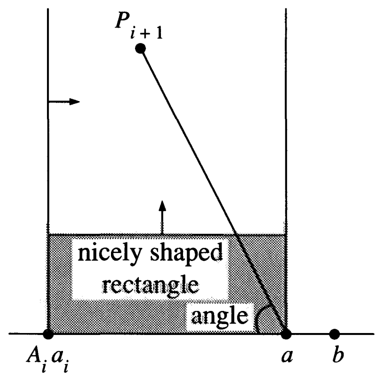


FIGURE 4

Case 2. There is a first i so that $p_i \in U_i$. At this point we may assume the angle as viewed from a is large using Case 1 and the fact that p_2 is to the right or on the vertical line at a. Then one applies a fixed number l of the factors A_iC_i, $A_{i+1}\mathbb{C}_{i+1}$, ... until a_{i+l} is much farther away from a than p_{i+l}. This happens because of assumption (ii) and statement (i) of the proof in case p_i lies in the Poincaré ball (Figure 2) of scale p_i. (The contrary case can be reduced by Koebe and induction to this case, and we leave this for the reader.) During these l iterations the angle as viewed from a is only boundedly distorted elementary geometry and the Remark below show. After that, the subsequent factors A_jC_j for $j > i + l$ only cause a sequence of distortions decaying geometrically by assumption (ii) (see Remark below). Thus, the angles of $p_2, p_3 \ldots$ remain large in all cases.

REMARK. If a holomorphic mapping is schlicht on $\mathbb{C} - \{x$ real but not in an interval $I\}$, then it has bounded distortion on any region as R_0 in Figure 5 on next page. The constants depend on the geometry of R_0. Also, it has exponentially small nonlinearity on a region such as R_n which is exponentially small.

Applying the first remark l times (l fixed) yields a bounded distortion in the above paragraph. Applying the second yields the geometric series of distortions used above. Q.E.D.

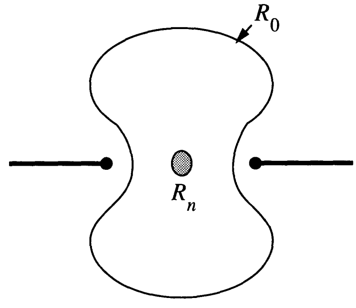


FIGURE 5

6. The factoring of the sector theorem. Consider an infinitely renormalizable mapping $f: I \to I$ of combinatorial type $\leq T$ with critical point c and critical value v. Let $\mathscr{C}_1, \mathscr{C}_2, \dots, \mathscr{C}_n, \dots$ be the interval collection at each level. Let $I_j(c) \in \mathscr{C}_j$ and $I_j(v) \in \mathscr{C}_j$ denote the intervals containing the critical point c and critical value v respectively (Figure 1). Assume $f = hQ$, where h^{-1} has a complex analytic injective extension to $\mathbb{C} - \{x$ real not in $J \supset I\}$ and Q is a quadratic polynomial, i.e., f belongs to the Epstein class (§4).

Consider the basic backwards composition $f(n)$ from $I_n(c)$ to $I_n(v)$ passing through each nth level interval in $\mathscr{C}_n$. Define the *scale of a factor* f^{-1} of $f(n)$ to be the largest s so that its domain interval at level n belongs to $I_s(v)$.

Divide the composition $f(n)$ into epochs by: epoch $n-1$ is from the beginning of the composition up to and including the last factor of scale $n-1$, epoch $n-2$ is from there up to and including the last factor of scale $n-2$, etc.

In epoch j mark all the left intervals of scale j, where a left (right) interval of $\mathscr{C}_n$ is one dynamically related to $I_n(v)$ by an orientation reversing (preserving) map. A left (right) root is by definition the part of the factor f^{-1} starting at a left (right) interval corresponding to Q^{-1} in the factoring $f = hQ$. Let C_α denote the part of basic composition between two marked left roots.

The backwards composition from $I_s(c) \to I_s(v)$ at level s restricted to an nth level interval in $I_s(c)$ is called a *basic map at level* s.

PROPOSITION. *Suppose the marked left root just after some composition C_α has scale r. Then C_α is a finite composition of basic maps at level r, right roots at scale r, and restrictions of h^{-1}. The number of each is bounded in terms of the bound T on the combinatorics.*

PROOF. *Claim*: The last visit during epoch j to an nth level interval in $I_j(v)$ lands on a left interval. (This useful observation was made by Wellington de Melo.)

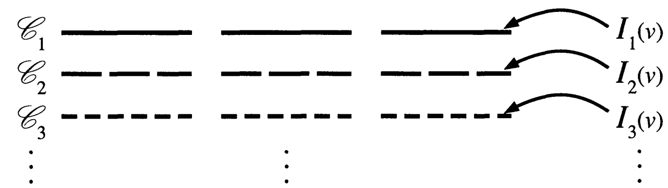


FIGURE 1

PROOF OF CLAIM. Consider the jth renormalization f_j of f preserving $I_j(v)$. Now f_j is unimodal so it maps $I_n(v)$ to the interval J_n furthest to the left in $I_j(v)$ by an orientation reversing mapping. Thus, J_n is a left interval and we see that under the backwards composition starting at $I_j(c)$ the inverse branches of f_j run through all the intervals of $\mathscr{C}_n$ in $I_j(v)$ arriving last at J_n and then going on to $I_n(v)$. This proves the claim.

Now by construction each epoch j is decomposed cleanly into basic maps of level j and single factors f^{-1} starting at intervals of scale j. By the claim the C_α either run between two left factors at scale j or start just after the last visit to $I_j(v)$ and run to the first left factor at scale $j-1$. In either case we have the structure required by the proposition. Q.E.D.

COROLLARY. *C_α satisfies property* (iii) *of the sector theorem relative to the immediately following left root.*

PROOF. We use the above proposition. Let a_i be the center of the left root following C_α at scale r. In §5 we have linearly renormalized all factors to fix $\{a, b\}$ and $|a-b| = 1$. In these terms the right roots of C_α at scale r are to the right of (a, b) at distance commensurable to $|a_i - a|$ because of bounded geometry of the Cantor set. Also, by the bounded structure of basic maps at level r these have bounded distortion between intervals containing the dynamic intervals at level r with space on either side. Thus, the region of control for each basic map covers the interval (a_i, a) plus space on either side because a_i is in the dynamic interval at scale r.

The fixed number of *right* roots, the fixed number of h factors, and the fixed number of basic maps only disturb this control a fixed amount. Q.E.D.

Finally we make explicit the connection between the marked left roots here, the C_α between, and the factoring

$$A_n C_n \cdots A_2 C_2 A_1 C_1$$

of the sector theorem (§5). We let A_1 be the marked left root at scale $n-1$ closest to $I_n(v)$. We let A_2, A_3, etc. be the subsequent marked left roots. Finally, the C_α are the in between compositions, as expected.

PROPOSITION. *Assume $f = hQ$ is of the Epstein class. Then the above composition satisfies hypotheses* (i), (ii), *and* (iii) *of the sector theorem, at level n.*

REMARK. The dependence of the constants (k, K, λ) on $f = hQ$ and n is "beau" as in §3.

PROOF. (i) is true by definition. (iii) is the above proposition. (ii) follows directly from bounded combinatorics and bounded geometry of the Cantor set.

We note only for (ii) that there are always left roots of scale j in epoch j (see the proof of the claim above).

Also, bounded combinatorics controls the number of marked left roots in epoch j because they lie in disjoint intervals at level j. Q.E.D.

7. The sector inequality. Suppose $c < c' < b' < b < a$ in the reals and F is a schlicht mapping of a hemidisk D of radius R and center b' to a sector with angles $(\theta, \pi/2)$ resting on (a', b') in the upper half plane H.

Let N be the geodesic neighborhood of (a, c) in H corresponding to the Euclidean disk of radius $\overline{R}$ whose boundary passes through $\{a, c\}$ (Figure 1). Assume θ is fixed, all nonzero distances between $\{a, b, c, a', b', c'\}$ are of order 1, and F carries $\{a, b, c\}$ to $\{a', b', c'\}$ in order.

THEOREM. *For $\overline{R}$ sufficiently large and $R/\overline{R}$ sufficiently large compared to $\overline{R}$, N contains $F(N)$ plus all the points of the upper half plane within a definite Euclidean distance to $F(N)$.*

PROOF. (1) Let ψ be the Riemann mapping of the upper half plane to the sector carrying (∞, a, c) to (∞, a', c'). Let $U(R)$ denote $\psi(D(R))$, where $D(R)$ is the hemidisk of radius R centered at b'.

(2) If U is a simply connected domain with arc γ on its boundary and F is a complex analytic mapping of $U \to U$ which is continuous at γ and preserves γ, let $\widetilde{U}$ and $\widetilde{F}$ denote the double of U and F along γ. Then γ is a geodesic in the Poincaré metric of $\widetilde{U}$ preserved by $\widetilde{F}$. By Schwarz's lemma $\widetilde{F}$ preserves the neighborhoods of γ of Poincaré distance $\leq c$ for all $c > 0$.

Thus, F preserves these regions intersect U. Abusing language we call them Poincaré geodesic neighborhoods of γ in U (see Figure 2).

(3) The Riemann map ψ, outside a fixed neighborhood of the corners (a, b, c) and $a', b', c')$, is the composition of a fractional power of z, z^{α}, and a mapping of uniformly bounded distortion of the Euclidean metric.

We will comment on the geometry ignoring this map of bounded distortion. For example, $U(R)$ is the intersection of the sector with the hemidisk of center b' and radius R^{α} (here $\alpha = 1/2 - \theta/\pi$).

(4) Our schlicht mapping F is then the composition of this power law and some schlicht mapping G of $U(R)$ into the sector which fixes $\{a', c'\}$.

(5) Let γ be the arc of the boundary of the sector from a' to c'. Then by the argument of (2) G takes the Poincaré geodesic neighborhoods of γ

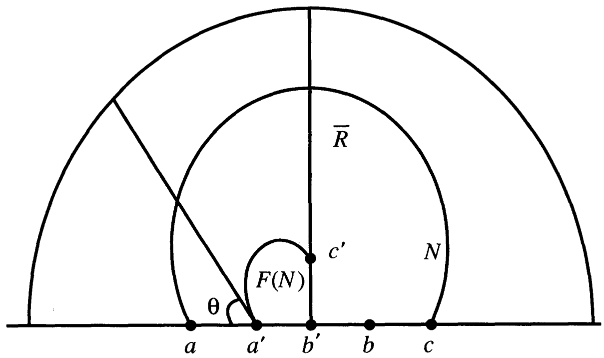


FIGURE 1

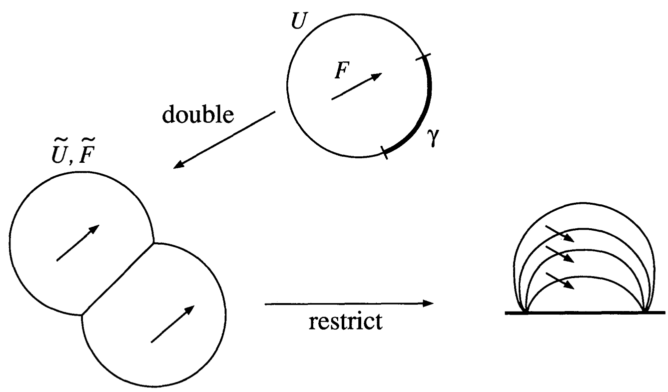


FIGURE 2

in $U(R)$ into the Poincaré geodesic neighborhoods of γ in the sector S preserving or decreasing "distance to γ."

(6) The boundary of S and boundary of $U(R)$ only start to differ at Euclidean distance R^α from γ. Thus, the geodesics much closer to γ than R^α are about the same for $U(R)$ or for S. So G does not move these neighborhoods too much. (To see this one can use the fact that the Poincaré metric of a simply connected plane domain is comparable to the Euclidean metric times the reciprocal of the Euclidean distance to the boundary.)

(7) Now a fairly large geodesic neighborhood N of (c, a) in the upper half plane is carried by ψ *well within* the sector (using (3) and our lower bound in the angle θ) to a geodesic neighborhood of $\gamma = (c', a')$ in the sector. By (6) G does not move it very much. Thus, the composition F does what we want in the geodesic neighborhood N corresponding to the circle of radius R so large that the power law beats the bounded distortion part of (3). Q.E.D.

Modified sector inequality. Replace the hemidisk D of radius R in the previous section by the largest "Poincaré neighborhood" M of (c, a) in the upper half plane contained in D (see the above proof). Suppose F is a schlicht mapping of M into the sector which is continuous on (c, a) and carries $\{c, b, a\}$ in order to $\{c', b', a'\}$. Let N as above be the Poincaré neighborhood of (c, a) of Euclidean diameter $\overline{R}$.

MODIFIED THEOREM. *For $\overline{R}$ large and $R/\overline{R}$ sufficiently large compared to $\overline{R}$, N contains $F(N)$ plus all the points of the upper half plane within a definite Euclidean distance to $F(N)$.*

PROOF. Let $\infty \in \partial M$ be the highest point of M. Let $\overline{\psi}$ be the Riemann mapping of M to the sector carrying $\{\infty, c, a\}$ in order to $\{\infty, c', a'\}$. As R approaches ∞, $\overline{\psi}$ approaches the Riemann mapping ψ of the previous proof uniformly on N since $\overline{R}$ is fixed. Then the proof above can be modified by continuity considerations to work here.

8. The complex quadratic-like mapping produced by renormalization. Let us work with renormalizable mappings $f = hQ\colon I \to I$ of combinatorics $\leq T$ of the Epstein class $E(J)$, $I \subset J$. After some renormalization we can assume J contains a definite neighborhood of I and that the real bounds on the critical orbit hold §§3, 4.

THEOREM. *For any $n > N(T)$ the nth renormalization $g = R^n f$ has a complex analytic extension G to some disk $D \subset \mathbb{C}$ so that $D \to G(D)$ is proper of degree two and $G(D) - D$ has conformal modulus $> m(T) \geq 0$ (see Figure 1).*

PROOF. Let $f(n)$ denote the backwards branch going from the critical point interval $I_n \in \mathscr{C}_n$ to the critical value interval $I_n v \in \mathscr{C}_n$.

Let (a, c) denote the maximal interval where $f(n)$ is a diffeomorphism into the reals and let $(\overline{a}, \overline{c}) = f(n)(a, c)$. Let $\{\pm a, \pm(c')\}$ be $f^{-1}\{a, c\}$.

By the sector theorem and §6 the image by $f(n)$ of the upper half plane is in the sector $0 \leq \arg(z - \overline{a}) \leq \pi - \theta(T)$ since we have arranged that J contains a definite neighborhood of I and the real bounds on the critical orbit hold.

Now apply two branches of f^{-1} to the sector (obtaining Figure 3(a), and (b)) applied to any geodesic disk on (a, c) of radius less than that of one contained in the region of bounded nonlinearity of h^{-1} (e.g. $\leq$ scale of J).

There are several points to make.

(1) It is simple to see that $g_+^{-1}M$ in Figure 3(a) lies in a sector $\pi/2 \leq$

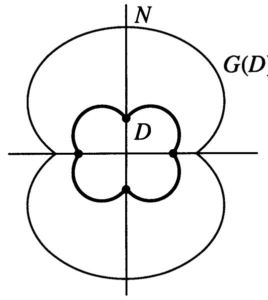


FIGURE 1

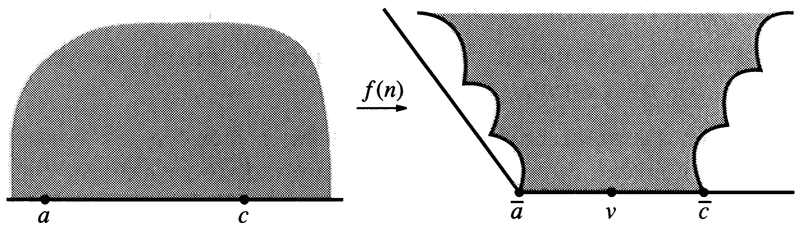
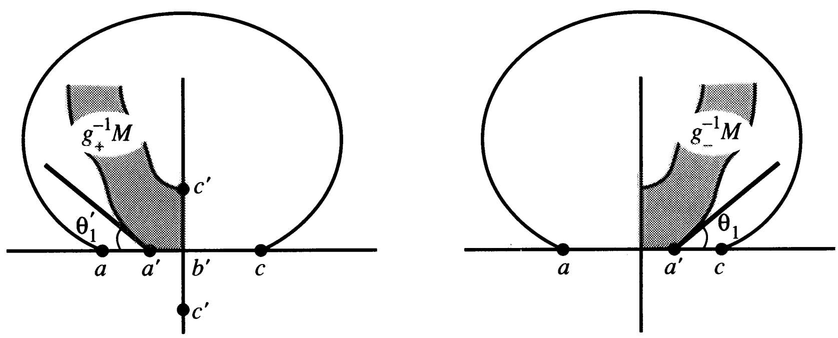


FIGURE 2



(a) (b)

FIGURE 3

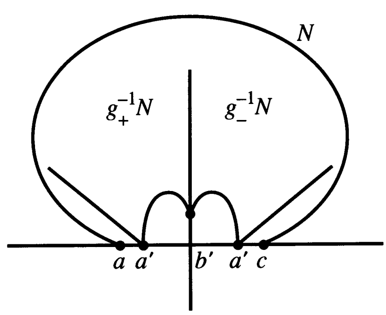


FIGURE 4

$\arg(z - a') \leq \pi - \theta_1(\theta_1 < \theta)$ because we have only applied a bounded distortion map h^{-1} to a piece of the sector in Figure 2 on previous page and then a right root. Here $g_+^{-1} = f_+^{-1} f(n)$.

(2) The same remark applies to $f_-^{-1} f(n) = g_-^{-1}$ and $g_-^{-1}(M)$ (Figure 3(b)).

(3) By the proposition below, distance(c, b') and distance(a, b') are each a definite factor greater than distance(a', b'), where b' is the critical point of $R^n f$.

(4) If $n(T)$ is large enough, then M can be taken large relative to N in the sector inequality for some $N \subset M$ with N large relative to (a, c). This yields Figure 4 on previous page.

Now we reflect across the x axis to obtain Figure 1 and the result. Q.E.D.

PROPOSITION. *Distance*(a, b') *and* distance(b', c) *are each greater than* distance(a', b') *by a definite factor.*

PROOF. We use what de Melo calls the basic fact, (a, c) contains the critical point interval $I_n \in \mathscr{C}_n$ and its two immediate neighbors. Call these the *three intervals.* Also a' is a critical point of f^{q_n}, where $R^n f = f^{q_n}/I_n$ which is closest to b'. Thus, a' lies in one of the preimages of I_n by f^{q_n} which is closest to I_n. The proposition follows from the bounded geometry of the *three intervals* and their critical points. This follows using §3 in the manner of the self-contained first paragraph of §15.

9. Douady-Hubbard theory and Riemann surface laminations. Consider the set $\mathbb{Q}_{\mathbb{C}}$ of complex quadratic-like mappings $F: D \to FD$ with connected invariant set $K_F = \bigcap F^{-n} D$ up to $\mathbb{C}$-analytic conjugacy near K_F. Two F, G are h-equivalent **[DH1]** if they are qc conjugate near K_F and K_G by a map which has no conformal distortion (i.e., $\bar{\partial}$ map =0, a.e.) between K_F and K_G a.e. Lebesgue.

THEOREM (Douady-Hubbard). (a) *The quadratic polynomials with connected invariant set cut each h-equivalence class in one point.*

(b) *Each h-equivalence class is bijectively equivalent to the set of real analytic degree two expanding maps of S^1 up to real analytic equivalence.*

REMARK. The quadratic polynomial in part (a) is called the internal class of $q \in Q_{\mathbb{C}}$ while the expanding mapping in part (b) is called the external class of $q \in Q_{\mathbb{C}}$. Thus the internal class is a point on the Mandelbrot set. We refer to the various "submanifolds" of $Q_{\mathbb{C}}$ with constant internal class, i.e., constant label in the Mandelbrot set, as the *prestable manifolds* of renormalization. In any $\mathbb{C}$-analytic family, fixing the internal class defines a $\mathbb{C}$-analytic subspace **[DH1]**. Therefore in the context of *symmetric* $\mathbb{C}$-analytic mappings the stable manifolds of renormalization (Theorems 2 and 2′) have at least a real analytic structure.

Now we associate to any *smooth* degree two expanding map f of the circle

($f' > 1$, f' Hölder) a Riemann surface lamination (appendix) whose point in Teichmüller space determines the smooth conjugacy class of f. For real analytic f the point determines the external class.

Form the inverse limit space, a dyadic solenoid,

$$\overline{S} = \varprojlim \{\cdots \to S \xrightarrow{f} S \xrightarrow{f} S\}$$

with induced mapping $\tilde{f} = \varprojlim\{f\}$. By the theorem below leaves of $\widetilde{S}$ carry unique affine structures compatible with their smooth structure so that $\tilde{f}$ becomes affine. Now attach upper half spaces of the leaves of $\widetilde{S}$ using these affine structures to obtain a Riemann surface lamination $\widetilde{L}$. Then $\tilde{f}$ on $\widetilde{S}$ extends affinely to a holomorphic mapping $\widetilde{F}: \widetilde{L} \to \widetilde{L}$. Remove the boundary and quotient by the group generated by $\widetilde{F}$ to obtain $L_f = \widetilde{L}/\{\widetilde{F}\}$.

The Riemann surface lamination L_f up to Teichmüller equivalence (appendix) remembers the conformal structure on leaves up to qc isotopy. Lifting the $\widetilde{L}$ and recalling $\widetilde{F}^{-1}$ is contracting we see the qc isotopy converging to the identity on the ∂ solenoid. Thus, the affine structure on the leaves is determined by L_f up to Teichmüller equivalence. (The unique field of affine structures on the dynamic solenoids $(\widetilde{S}, \tilde{f})$ are continuously varying. Thus, the structure on even one leaf determines the field of structures.)

The eigenvalues of f can be read off from the affine expansion factors of the $\tilde{f}$ periodic leaves. The eigenvalues of f determine the sizes (up to a bounded factor) of the intervals in the nth level Markov grids f^{-n} (fixed point of f). The Markov grids for f and g determine a unique conjugacy h which is quasisymmetric in general and obviously Lipschitz if the corresponding sizes are in bounded ratio. A Lipschitz conjugacy h between f and g is differentiable at most points. It follows that h' exists and is Hölder using the approximate formula $h \sim g^n Df^{-n}$ when f^{-n} is chosen to converge to a point of differentiability of h. When f and g are real analytic, domains of analyticity are easily controlled and h is seen to be real analytic.

Here is the discussion for the theorem below. The natural projection $\widetilde{S} \xrightarrow{\pi} S$ is used to induce smooth structures and smooth Riemannian metrics in the one-dimensional leaves of $\widetilde{S}$. The induced mapping $\tilde{f}$ on $\widetilde{S}$ then inherits the same geometric structure as f had. In particular, $\tilde{g} = \tilde{f}^{-1}$ is uniformly contracting on leaves. To construct a leaf translation x to y on the same leaf, approximately, we apply g^n until these points become $\overline{x}$ and $\overline{y}$, very close together. We translate $\overline{x}$ to $\overline{y}$ using the smooth structure and transport this back by g^n. The limit of this construction defines the translation pseudogroup of the affine structure on each leaf. It is invariant by construction. The construction also shows any invariant structure must be this one. The field of translation pseudogroups varies continuously because we have an a priori estimate on the error of the g^n approximation. This proves

THEOREM (global linearization). *Each leaf of $\widetilde{S}$ carries a translation pseudogroup or affine structure compatible with its smooth structure so that $\tilde{f}$ between leaves is affine. The affine translations in the leaves vary continuously in the transverse direction, and they are uniquely specified by continuity and $\tilde{f}$ invariance.*

COROLLARY. *The external classes $\{f\}$ can be embedded $f \to L_f$ in the Teichmüller space of a surface lamination (appendix).*

REMARK. In the real analytic case there is a direct construction of L_f and the linearization in terms of a holomorphic extension F of f to some neighborhood $F^{-1}U$ of $\{|z| = 1\}$ in $\{|z| > 1\}$. It is:

Form $\widetilde{V} = \varprojlim\{\cdots F^{-2}U \to F^{-1}U \to U\}$ and $\widetilde{F}\colon \widetilde{W} \to \widetilde{V}$, where $\widetilde{W} = \varprojlim\{\cdots F^{-2}U \to F^{-1}U\}$. This construction of passing to the inverse limit coverts F, which is neither globally defined on its image or injective, into $\widetilde{F}$, which is injective but still not globally defined on its image. So form $G = \widetilde{F}^{-1}\colon \widetilde{V} \to \widetilde{V}$, which is globally defined and injective but not onto. Now make G onto by passing to the direct limit $\widetilde{U} \to \varinjlim\{\widetilde{V} \to \widetilde{V} \to \cdots\}$ and $\widetilde{G} = \varinjlim G$ to obtain the bijection $\widetilde{G}\colon \widetilde{U} \to \widetilde{U}$. (The direct limit step was suggested by Lyubich.)

The leaves of the Riemann surface lamination $\widetilde{U}$ are upper half planes permuted holomorphically by $\widetilde{G}$. This proves anew the linearization theorem. The quotient of $\widetilde{U}$ be the group generated by $\widetilde{G}$ is the orbit lamination L_f.

10. The modulus function on external classes. Let $m(x)$ denote the supremum of the conformal modulus of N for some representative $F\colon F^{-1}N \to N$, where x denotes th $\mathbb{C}$-analytic equivalence class of germs of degree two holomorphic mappings of a neighborhood of $\{|z| = 1\}$ in $\{|z| \geq 1\}$. Let distance(x, y) denote the infimum of qc distortion of conjugacies between representatives on some neighborhood of $\{|z| = 1\}$ in $\{|z| \geq 1\}$.

THEOREM. *If $m(x) = \infty$, then x is represented by $z \to z^2$. If x and y satisfy $m(x) \geq \varepsilon$ and $m(y) \geq \varepsilon$ then* distance$(x, y) \leq d(\varepsilon) < \infty$.

PROOF. Choose coordinates so N is a standard annulus. The modulus of $F^{-1}N$ is $\frac{1}{2}$ modulus N. Thus $F^{-1}N$ contains a definite neighborhood of $|z| = 1$ in $|z| \geq 1$. We can apply Koebe distortion to control the nonlinearity of F. Then by replacing N by a smaller concentric annulus N' we have on $F^{-1}(N')$ a completely controlled analytic mapping. In particular, the geometry of the glued fundamental domain is controlled. Thus, for two such maps we can construct by pull back a quasiconformal conjugacy (§11) with controlled distortion. This proves the second statement. The first statement is standard.

11. Thurston equivalences and the pull back conjugacy. Start with two complex quadratic-like mappings $F: D \to F(D)$ and $G: D \to G(D)$. A *Thurston equivalence* between F and G is a certain kind of homotopy class of pairs

$$(X_F, C_F) \xrightarrow{H} (X_G, C_G),$$

where X_F is a contractible region containing the positive critical orbit $C_F = \{1, 2, 3, \dots\}$ of F, X_G is a contractible region containing the positive critical orbit $\{1, 2, 3, \dots\}$ of G, $H(1) = 1$, $H(2) = 2$, etc. and the phrase "homotopy class" is defined by isotopies of the contractible regions X_F, X_G fixing $\{1, 2, 3, \dots\}$ and homotopies of restrictions of H's to common smaller regions also fixing $\{1, 2, 3, \dots\}$.

To define which homotopy classes are Thurston equivalences consider Figure 1.

Start with a homotopy class H_0 and lift it through the branch cover so that 1 goes to 1. We *assume* (a) the lift H_1 carries 2 to 2, 3, to 3, etc. and (b) the homotopy class of H_1 equals the homotopy class of H_0 (rel $\{1, 2, 3, \dots\}$). In (b) we have used the (dynamic) point that the covering spaces are subsets of their respective bases. Note also that property (a) only depends on the homotopy class of H_0. *In effect we have defined a Thurston map on homotopy classes $H_0 \mapsto H_1$ and a Thurston equivalence is a homotopy class which is fixed by this Thurston map.*

For the *pull back conjugacy theorem* we start with a certain representative of a Thurston equivalence $H_0: F(D) \to G(D)$ which (a) is also a conjugacy between the maps $F: \partial D \to \partial F(D)$ and $G: \partial D \to \partial G(D)$ and (b) is a quasiconformal homeomorphism.

Then $H_0: F(D) \to G(D)$ and the pull back $H_1: D \to D$ restrict to maps $\partial H_i: \partial D \to \partial D$ satisfying $G \cdot \partial H_i = \partial H_0 \cdot F$, $i = 0, 1$, so $\partial H_0 = \partial H_1$ or $\partial H_0 = \partial H_i \cdot \tau$, where τ is the involution of the double cover $F: \partial D \to \partial G(D)$. If we lift the homotopy asserting H_0 and H_1 have the same homotopy class rel $\{1, 2, 3, \dots\}$, then we see the first possibility holds (Figure 2 on next page), i.e., $\partial H_0 = \partial H_1 : \partial D \to \partial D$.

Thus, we may add to H_1 the restriction of H_0 to the outer annulus $F(D) - D$ to get an extension of H_1 to a homeomorphism between $F(D)$ and $G(D)$. This homeomorphism, called $H_1: F(D) \to G(D)$, will be quasiconformal if we make the technical assumption that ∂D in $f(D)$ is a quasicircle. Its

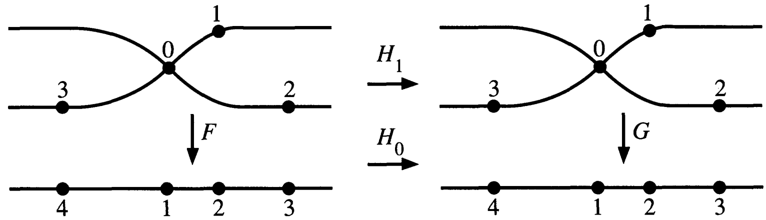


FIGURE 1

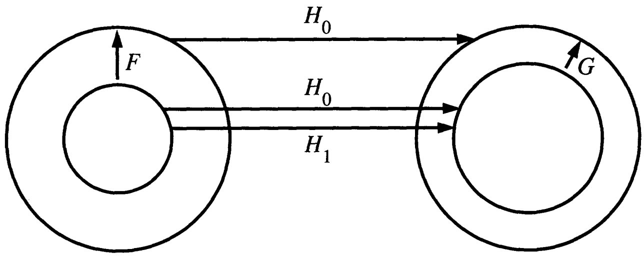


FIGURE 2

distortion will be the same as that of H_0 because it is the union of a piece of H_0 (on $F(D) - D$) and a "complex analytic conjugate" of H_0 (since $GH_1 = H_1 F$ on D). (This step was suggested by Curt McMullen.)

We iterate the process to construct an infinite sequence of K-quasiconformal homeomorphisms with fixed values on $\{1, 2, 3, \dots\}$ and not changing on the outer rings $F(D) - D$, then $D - F^{-1}D$, then $F^{-1}D - F^{-2}D$, etc. By *quasiconformality*, we can extract convergent subsequences on all of $F(D)$. The limit actually exists on the union of the outer rings. So if this union is dense all limits over subsequences are equal and equal to the continuous extension of what is on the outer rings. If the union is not dense there is some ambiguity in the limit on the interior of the invariant set. This interior is classified in types by Sullivan [S5] and homotopies to conjugacies and discussed in Mane-Sad-Sullivan [MSS] and McMullen-Sullivan [MS] (see also [ST]). We continue this discussion in the more rigid case when there is no interior and the limiting map in unique and is a conjugacy.

The conformal distortion of the limiting conjugacy depends on the distortion of H_0 on the fundamental domain and the distortion on the Julia set. The latter may be zero by construction as in §14 where internal classes are discussed. Or it may be zero because the Julia set may not support a new measurable invariant conformal structure. An example of the latter is the infinitely renormalizable bounded type symmetric complex quadratic-like mapping (Theorem 6, §15). We express this last property by there is no "measurable invariant line on the Julia set" or "no measurable line field." All the above amounts to the

THEOREM. *Suppose F and G are complex quadratic-like mappings which are Thurston equivalent* (*along their positive critical orbits*) *via a quasiconformal homeomorphism. Then if $\partial D \subset F(D)$ and $\partial D \subset G(D)$ are quasicircles there is a quasiconformal conjugacy $H\colon F(D) \to G(D)$ between F and G. The conformal distortion of H depends only on the pairs $(F(D) - D, F/\partial D)$ and $(G(D) - D, G/\partial D)$ if the invariant set has no interior and the Julia set has no "measurable line field."*

REMARK. Otherwise the distortion depends also on the construction of the conjugacy on the invariant set. (a) In case there is no interior for the invariant set but the Julia set has a "measurable line field," the construction is canonical and leads to a specific distortion on the Julia set parametrized by a complex number in the unit disk (see **[MSS]** for more details, however conjecturally this case does not exist). (b) In case there is interior there must be either (i) a super attracting cycle, (ii) an attracting cycle, (iii) an indifferent periodic point, or (iv) a Siegel disk **[S5]**. In cases (i) and (iii), the construction can be made so that the limit has no conformal distortion on the filled in Julia set **[MSS]**. In case (ii) the distortion required is essentially $|\log \lambda_1/\lambda_2|$, where λ_i are the eigenvalues at the attracting cycle **[MSS]**. In case (iv) it is reasonable to think the Thurston equivalence makes the eigenvalues equal and then the construction can be made with no distortion on the filled in Julia set (see **[MSS, ST, S5]**). This point does not concern us in this paper where we study mappings symmetric about the real axis.

PROOF. Besides the discussion before the statement of the theorem we need to add that a quasiconformal homeomorphism representing a Thurston equivalence can be restricted, deformed by an isotopy, and extended to fit with a quasiconformal equivalence between the pairs $(F(D) - D, F/\partial D)$ and $(G(D) - D, G/\partial D)$ to satisfy (a) and (b) of the second paragraph of this section. This is elementary planar topology. Q.E.D.

12. Renormalization of complex quadratic-like mappings. One says a complex quadratic-like mapping F with connected invariant set is *renormalizable* if there is an n and a disk D containing the critical point so that $F^n: D \to F^n D$ is complex quadratic-like with connected invariant set. Let $RM \subset M$ be the subset of points of the Mandelbrot set M which have representative quadratic-like mappings which are renormalizable. Recall the Douady-Hubbard theorem (§9) that germ equivalence classes of quadratic-like mappings are isomorphic to $M \times T = \{\text{internal class, external class}\}$. We can define a renormalization operator R on representatives by $F \to F^n/D$, where n is minimal.

THEOREM. *Renormalization defines a mapping* $RM \times T \xrightarrow{R} M \times T$ *respecting the prestable manifolds* $\{\text{pt} \times T\}$, *namely* R *equals the union over* RM *of mappings* $\{m, T\} \xrightarrow{R_m} \{R(m), T\}$ *for an induced surjective operator* $RM \xrightarrow{R} M$. *The individual mappings* R_m *are induced by mappings on the spaces of special conformal structures preserving special Beltrami paths.*

DEFINITION. A *special Beltrami path* is one coming from a Beltrami path of invariant conformal structures outside the filled in Julia set. The latter are the *special conformal structures.*

PROOF. The theorem follows, once $m \in M$ is fixed, from the definitions and the picture of renormalization, as merely the restriction of the variable conformal structure exterior to the Julia set of f to the exterior of the Julia set of $Rf = f^n/D$.

Addendum. (Renormalization of vector fields on the laminations.) A backward orbit of G, the renormalized mapping of F, extends in a natural way to a backward orbit of F. Thus we have an embedding of L_G, the lamination of G, minus the contribution $\widetilde{K}_F$ coming from the germ of K_F near K_G, in L_F, the lamination of F. Thus a continuous vector field on L_F with $\overline{\partial}$ in L^∞ can be restricted to obtain one on $L_G - \widetilde{K}_F$ which has $\overline{\partial}$ in L^∞ and is uniformly bounded in the P-metric coming from (nghd of $L_G - \widetilde{K}_F$). Extending by zero on $\widetilde{K}_F$ we obtain a continuous vector field on L_G with $\overline{\partial}$ in L^∞. (This may be shown most easily using the equivalence ($\overline{\partial}$ in L^∞) $\sim$ (Zygmund in conformal metric).)

13. Teichmüller contraction of renormalization for symmetric complex quadratic-like mappings. If a real analytic folding mapping f has a complex quadratic-like extension we have two notions of renormalization—real renormalization (of the introduction and §§1–4) and complex renormalization (of §12). The following theorem is proved below.

THEOREM 13.1. *These two notions are compatible in the sense that whenever real renormalization is possible for f for some n, then for the same n so is complex renormalization for the complex extension F of f. Moreover, the complex extension of Rf is a quadratic-like mapping which is $\mathbb{C}$-analytically equivalent to RF on neighborhoods of their respective connected invariant sets.*

Let f have a complex quadratic-like extension $F: D \to F(D)$ so that the conformal modulus of $D - F(D) \geq \varepsilon > 0$. Suppose f admits $n \geq n(T, \varepsilon, l)$ renormalizations of return times $\leq T$ for a certain function $n(T, \varepsilon, l)$, where l is chosen below. Let μ be a Beltrami coefficient defined on $F(D) - D$ which is symmetric about the real axis and let $|\mu|_T$ denote the Teichmüller length of $\mu = \sup_{|\varphi|=1} \int \varphi\mu$ (see appendix). Let l' be determined by the universal bounds of §§8 and 10. Choose $l > l'$ and let $\lambda(l, l') < 1$ be determined by the Grötzsch inequality (see appendix). Then we have renormalization qua Teichmüller contraction.

THEOREM 13.2. $|R^n\mu|_T \leq \lambda(l, l')|\mu|_T$ *for* $n \geq n(T, \varepsilon, l)$.

COROLLARY. *Under bounded return time renormalization of symmetric Mandelbrot internal classes, the Teichmüller distance between external classes decreases exponentially fast.*

PROOF OF THEOREM 13.1. Now suppose for some n, g^n is renormalizable on the real axis, i.e., there is the little box on the diagonal of Figure 1(c) of the introduction which encloses the graph of Rg. Consider the connected component D_R in $G^{-n}U$ of the critical point in the complement of $G^{-2}\gamma, G^{-3}\gamma, \ldots, G^{-n}\gamma$. Then by construction G^n has one critical point inside D_R. The intersection J_R of D_R with the real axis is the interval

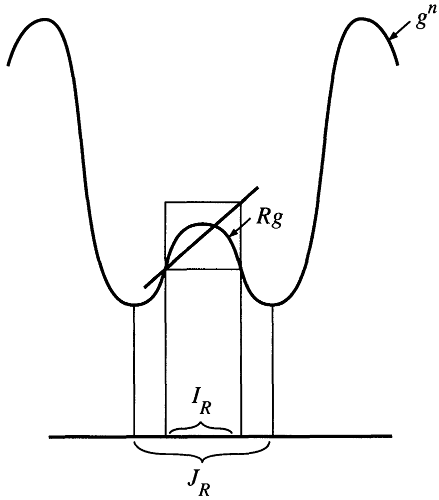


FIGURE 1

between the next two critical points of g^n out from the central one again by construction: thus, it contains the dynamic interval I_R of Rg (Figure 1).

The image of D_R by g^n is $U = \mathbb{C} - \{x \text{ real not in } J\}$ further slit in from either side by the images under G^n of the arcs $G^{-2}\gamma, G^{-3}\gamma, \ldots, G^{-n}\gamma$. These are $\gamma, g\gamma, \ldots, g^{n-2}\gamma$. By the discussion of §8, based on point (2) of the proof of Theorem 1 in §3, the unslit interval covers the small dynamic interval $[R_g(c), (R_g)^2c]$ and its two immediate neighbors in the nth level collection of small dynamic intervals (see Figure 3, §3). And these intervals cover J_R as discussed in §8. Thus, $G^n: D_R \to$ (U with further slits) is a quadratic-like mapping—in the 4-fold symmetric Epstein form.

PROOF OF THEOREM 13.2. We choose an extremal $\tilde{\mu}$, representing the same tangent vector to L as μ defined on the same fundamental domain (see Remark below). We deform along a Beltrami path a distance l greater than l' (see below). We may assume $\tilde{\mu}$ is also symmetric about the real axis. Then the two $\mathbb{C}$-analytic systems at the endpoints have a definite modulus (§10). So there is a definite number of renormalizations required so that the modulus is greater than the universal constant of §8. Then the qc distance is at most a certain constant, call it l' (§10). Now we apply the almost geodesic lemma (appendix) to see that the renormalized tangent vector is reduced in Teichmüller length by a universal factor $\lambda(l, l') < 1$.

PROOF OF COROLLARY. Choose a Beltrami path between two real analytic external classes. The tangent vectors along these paths are represented by μ's to which Theorem 13.2 applies. As we renormalize this continues to be true by the complex bounds (§8). Thus, the Teichmüller arc length of this path decreases exponentially by integrating the inequalities of Theorem 13.2.

REMARK. If a tangent vector to $T(L_f)$ is defined by an F-invariant μ on a neighborhood U of the circle, we can push forward the $\widetilde{F}$ invariant holomorphic quadratic differentials on $\widetilde{L}$ to obtain holomorphic quadratic differentials φ on U satisfying $F_*\varphi = \varphi$.

The holomorphic invariants (φ, μ) are just $\int \varphi\mu$ on any fundamental domain of F. By Hahn-Banach there is a $\tilde{\mu}$ on this domain with L^∞-norm $\tilde{\mu} = \sup(\varphi, \mu)$. This is the definition of an extremal $\tilde{\mu}$. By the appendix there is a vector field V on the lamination so that $\overline{\partial} V$ is $\mu - \overline{\mu}$ pulled up to the lamination. We can renormalize V (see Addendum §12) to see that $R^n\mu$ and $R^n\overline{\mu}$ pulled to the appropriate lamination differ by $\overline{\partial}$ (vector field). Thus they continue to have the same T-norm by the integration by parts formula (appendix). Alternatively (McMullen), we can see they continue to have the same T-norm by pushing forward quadratic differentials using the map of the Addendum §12.

14. Proof of Theorem 2′. Let $\{f_n\}$ be an inverse chain related by renormalization, $\cdots \to f_n \xrightarrow{\sigma_n} f_{n-1} \to \cdots \to f_2 \xrightarrow{\sigma_2} f_1 \xrightarrow{\sigma_1} f_0$, where σ_i is bounded in size by T. Then if all the f_i are smooth quadratic-like mappings which are uniformly bounded, by §4 they are all Epstein and by §8 they are all complex quadratic-like with a definite modulus (§10). Let $c(f)$ denote the internal class of Douady-Hubbard (§9). Suppose $\{g_n\}$ is another such inverse chain with the same combinatorics and suppose $c(f_0) = c(g_0)$. We want to show $\{f_n\} = \{g_n\}$ as complex analytic mappings up to affine rescaling.

(1) $c(f_n) = c(g_n)$.

PROOF OF (1). Start with a quasiconformal conjugacy between f_0 and g_0 expressing the fact that $c(f_0) = c(g_0)$ (§11). By a finite construction on the real axis then the two half planes, this can be promoted to a qc homotopy conjugacy between f_1 and g_1 which is a conjugacy between the forward critical orbits of f_1 and g_1 (§11).

Now perform the pull back conjugacy construction of §11 to obtain a quasiconformal conjugacy between f_1 and g_1 which is a.e. conformal on the saturation of the filled in Julia sets of f_0 and g_0. The measure of what is left in the filled in Julia set of f_1 or g_1 is zero (see Remark below). Thus, $c(f_1) = c(g_1)$. Continuing in this way $c(f_n) = c(g_n)$.

(2) Now work in the topology of uniform convergence on a definite neighborhood of the dynamic interval where all maps have the Epstein form and are normalized to have the same dynamic intervals. We will choose this neighborhood once and for all to include the Julia sets of all the maps appearing in inverse chains. If this were not possible the modulus bound on the annuli (§8) would be violated.

The closure K of these maps in this topology is compact because they are actually bounded in the space defined by sup norm on a larger neighborhood.

It is also clear that the set of elements of bounded inverse chains is closed.

(3) By Theorem 13.2 the Teichmüller distance d between f_n and g_n must be zero.

(4) Let (F, G) be a limit point of (f_n, g_n). Then F, G have the same internal class. This continuity of the internal class is proven in **[DH1]**. Since F and G are members of inverse chains we must also have $d(F, G) = 0$.

(5) Take a $\mathbb{C}$-analytic conjugacy between F and G on some neighborhood of the filled in Julia sets (see §9). This means for a sequence of $n \to \infty$ we can have K_n qc conjugacies between f_n and g_n on definite neighborhoods of filled in Julia sets and $K_n \to 1$.

Now we view (f_0, g_0) as lying deep inside (f_n, g_n) and take a limit of the above conjugacies. The fixed neighborhood of f_n becomes a huge neighborhood of f_0 and the limiting conjugacy is a $\mathbb{C}$-analytic conjugacy between f_0 and g_0 defined on all of $\mathbb{C}$. Q.E.D.

REMARK. McMullen offers this proof: It is known (Lyubich) that *almost every point in a Julia set of a complex quadratic-like mapping is forward asymptotic to the closure of the forward critical orbit.* If we apply this statement to f_1 we see almost every point in the Julia set of f_1 eventually lands on the Julia set of f_0 by the renormalization disk picture (§12).

From the classification **[S5]** the interior of the filled in Julia set of f_1 is the union of the preimages of the interior of the filled in Julia set of f_0.

15. Proof of Theorem 2.

15. Proof of Theorem 2. Theorem 2 follows from Theorem $2'$ and the following (see §9).

THEOREM 6. *If two symmetric complex quadratic-like maps F and G have the same bounded type $(\sigma_0, \sigma_1, \dots)$, then they have the same internal class.*

PROOF. All the renormalizations are bounded smooth quadratic-like mappings by Theorem 1. Because the combinatorics is bounded this means the geometry of the interval collections at one level inside an interval at the previous level is bounded. Otherwise, we could change the kneading sequence in a limit of bounded shape examples with converging combinatorics, and this would contradict the continuity of kneading sequences in the C^1 topology.

Bounded geometry of the interval collections means we can present the critical orbit Cantor sets as an intersection of symmetric pictures in a plane of disks within disks of bounded geometry (Figure 1 on next page). But then we can construct construct a qc mapping between two such critical orbit Cantor sets by choosing the standard symmetric rigid maps between corresponding circles and extending them to bounded distortion diffeomorphisms between intermediate "pairs of pants" regions. Using §11 we can promote this to a symmetric qc conjugacy between F and G.

The complex polynomial $z \to z^2 + c$, c real equivalent to F, admits a symmetric invariant conformal structure on its Julia set. This conclusion is valid for all the polynomials $z \to z^2+c$ with this kneading sequence (thinking

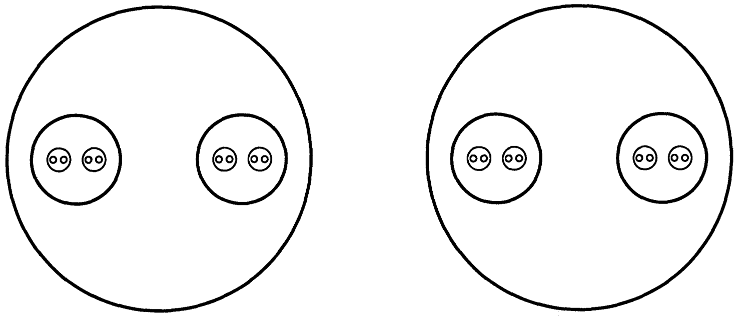


FIGURE 1

of them as G above, e.g.). There is a closed interval of such c's if Theorem 6 is false. Apply this statement about conformal structures to an endpoint of this interval. Construct a quasiconformal deformation of this quadratic polynomial Sullivan [S5] to see it is not the endpoint. Contradiction.

REMARK. The theorem is unknown for unbounded type because even though we have by Theorem 1 uniform bounds on the renormalizations we do not know there is a qs conjugacy between critical orbits of two f and g of the same unbounded type. This gap, because of Yoccoz's recent work, and the above Theorem 6 is the only one left to settle the celebrated question of density of hyperbolic systems in the quadratic family.

Appendix. Riemann surface laminations and their Teichmüller theory. Here is what we need to provide a basis for the argumentation of §§9–15.

(1) The notions of Riemann surface laminations, Beltrami tensor, and integrable quadratic differential have to be defined.

(2) (a) For a real analytic degree two expanding map f, L_f constructed as in §9 or as the space of orbits of $\overline{F}$ on $\widetilde{U}$, where $\widetilde{U} = \varprojlim\{\cdots F^{-2}U \to F^{-1}U \to U\}$, $\overline{F} = \varprojlim F$, and F is a $\mathbb{C}$-analytic extension of f to a neighborhood U, should be a Riemann surface lamination.

(b) The integrable quadratic differentials $\overline{\varphi}$ on L_f lift to $\widetilde{U}$ and then project to φ on U, which satisfy $F_*\varphi = \varphi$. (This means a system $\{\varphi_m\}$ with φ_m on $F^{-m}U$ and $F_*\varphi_m = \varphi_{m-1}$.)

(c) If a Beltrami tensor $\tilde{\mu}$ on L_f comes from an F-invariant Beltrami coefficient on U, then the pairing $(\overline{\varphi}, \overline{\mu})$ is computed as an integral $\int \varphi\mu$ over a fundamental domain of F in U.

(3) Two Beltrami tensors μ and γ on L_f differ by a trivial deformation $\mu - \gamma = \overline{\partial}V$ for a continuous vector field on L_f tangent to the leaves iff the holomorphic invariants are equal, $(\mu, \varphi) = (\gamma, \varphi)$, for all integrable

holomorphic quadratic differentials φ on L_f.

(4) The notion of holomorphic quadratic differential also has to be such that the Grötzsch inequality is valid. If ψ_t, $0 \le t \le 1$, is a quasiconformal isotopy in the leaves of the lamination between conformal structures c_o and c_l, where c_l is obtained by stretching c_o by a factor l along the trajectories of φ, then $l \le \int K(x)\,d|\varphi|$, where $K(x)$ is the conformal dilatation of ψ_1 at x.

(5) The Grötzsch inequality leads directly to the almost geodesic lemma. We say μ is ε-extremal if $|\mu|_\infty \le (\sup \int \varphi\mu)(1+\varepsilon)$, $|\varphi| = 1$. (Extremal means ε-extremal for $\varepsilon = 0$.) This allows one to prove the *almost geodesic* property of the global deformation determined by stretching a distance l along an ε-extremal μ. Let ψ_t, $0 \le t \le 1$, be a qc isotopy which compares the initial conformal structure c_o and the conformal structure c_l obtained by stretching along μ, which is ε-extremal, a distance l. Let K be the maximum dilatation of ψ_1, then there is a universal function $\delta(\varepsilon, l)$, where $\delta(\varepsilon, l) \to 0$ as $\varepsilon \to 0$ so that $l \le K(1-\delta)$ (*almost geodesic lemma*).

Here we go.

(1) and (2) A closed Riemann surface lamination L will be a compact space so that each point has a neighborhood (open disk $\times$ transversal) with overlap homeomorphisms $F(z, \lambda)$ preserving the disk factors and holomorphic in z. Beltrami tensors from the point of view of functional analysis are bounded Borel measurable functions modulo equality a.e. in each disk. We assume in addition that as a function of λ, $\mu(z, \lambda)$ varies continuously in the topology of convergence against each element in L^1 (disk). For changing coordinates, $\mu(z, \lambda)$ is the coefficient of the tensor $d\bar{z}/dz$. Quadratic differentials analytically, in a product chart, are elements in a direct system of L^1 spaces. The direct system is all σ-finite measure classes on the transversal. For each of these we form the product measure class with Lebesgue measure on the disk, form the L^1 space, and take the union (or rather direct limit). For changing coordinates these objects can be viewed for each measure as L^1 cross sections of the line bundle whose fiber is (volume elements of measure class on the transversal) $\otimes$ (dz^2 on disk).

We lose no generality by restricting attention to transversally invariant measure classes because these are cofinal in the directed set of all transversal measure classes.

We have a pairing (φ, μ) between Beltrami coefficients and quadratic differentials by integration over L. With these definitions the requirements of (2) are satisfied.

(3) (a) Using the formula on one disk

$$V = \int_D \mu \frac{d\zeta\, d\bar{\zeta}}{\zeta - z} + \int_{\partial D} V \frac{d\zeta}{z - \zeta},$$

where $\mu = \bar{\partial} V$ shows V is bounded near $O \in D$ with a modulus of continuity of the form $\delta \log \delta$ near there with constants depending on $|\mu|_\infty$ whenever V vanishes on ∂D.

(b) Now a continuous vector field V on a compact Riemann surface lamination L is bounded by compactness. We now make a hyperbolic assumption on L: there is a continuous leafwise Riemannian metric on L which is conformal and which on the universal cover of each leaf is uniformly quasi-isometric to the hyperbolic metric on the cover which is assumed to be the disk. Then V vanishes at the boundary of the cover and by (a) we have bounds and a leafwise modulus of continuity (in the hyperbolic metric and therefore the Riemannian metric) controlled by $|\overline{\partial} V|_\infty$.

(c) Suppose $\overline{\partial} V_i = \mu_i$ and $\mu_i \to \mu$ weakly in L^∞ on each leaf. By (a) and (b) we can in each leaf take pointwise limits of the formula (a) with no boundary term. The limit V satisfies $\overline{\partial} V = \mu$ and the formula with no boundary term for every point in each leaf. The argument works for nets as well as sequences.

Claim. V is a continuous vector field on L.

Proof of Claim. We combine four points

(i) The formula (a) can be restricted to a large metric disk in a leaf to compute V approximately at the center.

(ii) If $x_i \to x$ in L, large metric disks about x_i converge isometrically to a large metric disk about x (or at least a covering of such a metric disk). This is the basic property of laminations.

(iii) The kernel dz/z of formula (a) is almost fixed in L^1 by a mapping fixing O of a large hyperbolic disk with small isometric distortion.

(iv) Since μ is weakly continuous in the λ variable (by assumption) and L^∞ bounded, its integral against a λ-continuous family of L^1 functions is continuous in λ.

(d) By (c) the bounded μ's of the form $\overline{\partial} V$ for continuous V's on L is closed for the weak topology defined by integration against integrable quadratic differentials. Thus, these μ's are precisely those which annihilate all the φ's which annihilate all the $\overline{\partial} V$. But the integration by parts formula $\int_L \overline{\partial} V \varphi = \int V \overline{\partial} \varphi$ shows $\int V \overline{\partial} \varphi = 0$ for all V, or φ is holomorphic on leaves. This proves (3).

(4) A holomorphic quadratic differential can be viewed locally as a positive measure on the transversal times an L^1 function in the holomorphic quadratic differentials on the disks. With this convention in mind we associate to a holomorphic integrable quadratic differential φ (i) a metric on leaves on curvature ≤ 0 (the coordinates where φ is locally dz^2 or $z^k dz^2$), (ii) the measured lamination on each leaf whose trajectories are tangent to the line elements so that φ (line element) > 0 and whose transverse measure is determined by the metric of (i), and (iii) a further transverse measure to these trajectories in the transverse direction defined by the (transverse) measure defining φ.

These trajectories so transversally measured defined a generalized closed *geodesic* curve $|\varphi|$ which is of course tangent to the leaves. If we deform this curve by an isotopy ψ_t, $0 \le t \le 1$, in the leaves the *length can only increase* because the curvature is ≤ 0 and we start from a geodesic. Let c_o be the original conformal structure and let c_l be the conformal structure associated to the metric obtained by stretching the φ-metric by a factor l along the trajectories of φ.

Let $K(x)$ denote the conformal dilatation of ψ_1 between c_o and c_l. Let $J(x)$ be the Jacobian of ψ_1 between these two metrics. Then we compute the length of ψ_1 (curve) in the stretched metric

$$\left|\int D(x)\,d\varphi\right| \le \int K^{1/2}(x)J^{1/2}(x)\,d\varphi \le \left(\int K\,d\varphi\right)^{1/2}\left(\int J\,d\varphi\right)^{1/2},$$

where $D(x) \le K(x)^{1/2}J(x)^{1/2}$ is a derivative of ψ_1.

Assume $|\varphi| = 1$; then $\int J\,d\varphi = l$ and recall the length of the image can only be longer than the homotopic closed geodesic whose length is $l|\varphi| = l$. We deduce the Grötzsch inequality

(d) $l \le \int |K(x)|\,d|\varphi|$.

Remark. One can write an exact formula

$$D(x) = K^{1/2}(x)J^{1/2}(x)(\text{angle factor})^{1/2}$$

and get the better inequality (Reich Strebel)

$$l \le \int |K(x)\ (\text{angle factor})|\,d|\varphi|,$$

which implies if $\sup K(x) = l(1+\varepsilon)$ for ε small, then the angle factor must be near 1 on a fraction of points near 1 relative to the measure $|\varphi|$.

(5) Now we prove the almost geodesic lemma mentioned above. Suppose μ is ε-extremal and choose an integrable holomorphic quadratic differential φ of norm 1 so that $|\int \varphi\mu|(1+\varepsilon) \ge |\mu|_\infty$.

It follows by elementary arithmetic that μ must line up in measure (relative to $|\varphi|$) with the trajectories of φ, and (relative to $|\varphi|$) have essentially constant L^∞ norm. So stretch the original conformal structure in the μ direction by this essentially constant factor l to obtain c_l.

Now stretch the conformal structure c_0 by a factor l along φ, where $l = |\mu|_\infty$, to obtain $\bar{c}_l$. Consider the map ψ between $\tilde{c}_l$ and c_o which is $\bar{c}_l \xrightarrow{\text{identity}} c_l \xrightarrow{\psi_1^{-1}} c_o$. If K is dilatation of ψ_1 between c_l and c_o, then the dilatation of the composition above is at most $2l+K$ at almost all points and $\le K+o(1)$ at most points because μ and φ line up in measure relative to $|\varphi|$.

Now c_o is obtained from $\tilde{c}_l$ by stretching along the orthogonal trajectories by a factor l. By Grötzsch, $l \le \int |\text{dilatation}\,\psi|\,d|\varphi|$. This is a contradiction if $K = l'$ is less than l by an amount which is independent of ε for l and l' fixed.

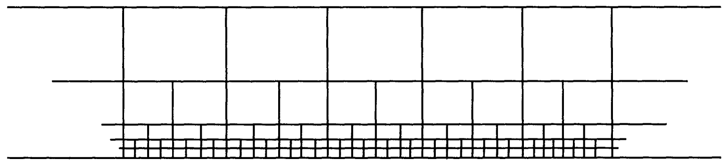


FIGURE 2

(6) The Teichmüller metric is defined infinitesimally by $\sup \int_L \mu\varphi$, φ holomorphic and $|\varphi| = 1$, for all Beltrami coefficients μ which define tangent vectors to the space of conformal structures on L. We integrate to get the length of paths of conformal structures and define the resulting metric on leafwise isotopy classes of conformal structures on a fixed background quasiconformal model L. This defines the *Teichmüller metric space of* L.

In the L_f case, f real analytic, the Teichmüller metric is related to the qc conjugacy metric by the following:

REMARK. All the f^{-n} have small ratio distortion at small enough scales, with constants depending on the Hölder constants. Similarly, all the F^{-n} for some $\mathbb{C}$-analytic extension F of f to a neighborhood U of $S = \{|z| = 1\}$ have small nonlinearity on a small enough neighborhood of S, with constants depending on U.

The T metric d between f and g estimates linearly log ratio descrepancies between the Markov grids for f and g at arbitrarily fine scales. (Proof: these ratios are organized by a Hölder continuous scaling function on the Cantor set of ends of the tree of inverse branches [S3] and d estimates the speed of change of the scaling function; (see example below). Thus, if $d(f, g) \leq \varepsilon$ and F and G are defined on definite neighborhoods U, then all corresponding consecutive ratios between the grids for f and g are estimated by $O(\varepsilon)$ below a scale depending on U and ε. Also, we can construct for F and for G an invariant system of vertices for a system of Carleson boxes starting at a definite scale (see Figure 2).

We divide each box into three triangles and construct an almost simplicial conjugacy between fundamental domains of F and G. We pull this back to obtain a qc conjugacy between F and G.

CONCLUSION. If the Teichmüller $d(f, g) = O(\varepsilon)$, f and g have $\mathbb{C}$-analytic extensions to a definite neighborhood U of $\{|z| = 1\}$, then on smaller definite neighborhoods (depending on ε and U) there is a qc conjugacy between F and G with conformal distortion, $O(\varepsilon)$.

EXAMPLE (quadratic differentials on L_f). An integrable holomorphic quadratic differential $\tilde{\varphi}$ on an upper half plane leaf of $\overline{U}$ can be pushed down to L_f. For example if we take $\tilde{\varphi}$ to have poles at infinity plus three points of the Markov grid of a solenoidal leaf (the grid is pulled up from the circle) we obtain a φ which measures the change in one of the asymptotic ratios which are recorded by the scaling function [S3].

REMARK. (1) The Teichmüller discussion above goes through for locally compact laminations. We only need to require that the quasiconformal vector fields are continuous and uniformly bounded. Then the proofs of (3), (4), (5), and (6) are unchanged. In this way we have a generalization of classical Teichmüller space by considering the case of a lamination with one leaf.

(2) There are enough integrable holomorphic differentials to have a version of (3) in the *measurable theory* where $\mu(x, \lambda)$ and $V(x, \lambda)$ are defined, uniformly bounded, and measurable in λ for each transversal measure class in a consistent manner. Then the Beltrami tensors form the complete dual of measurable integrable quadratic differentials.

The proof of (3) is the same and in addition there is the formula for the Teichmüller norm (measurable theory) $\inf_V |\mu + \bar\partial V| = \sup_\varphi \int \mu\varphi \equiv |\mu|_T$, φ holomorphic of mass one. We used this formula in the *continuous theory* for L_f for those tangent vectors μ coming from fundamental domains of F.

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