Proceedings of the AMS Centennial Symposium
August 8–12, 1988

From Quantum Theory to Knot Theory and Back: a von Neumann Algebra Excursion

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In the usual formulation of quantum mechanics, the pure states of a physical system are given by the one-dimensional subspaces of a complex Hilbert space $\mathscr{H}$, with inner product $\langle\ ,\ \rangle$. (If $\mathscr{H}$ is realized as a concrete L^2 space of functions, a function of unit norm in the one-dimensional subspace is called the wave function.) The observables of the system are given by selfadjoint (possibly unbounded) operators on $\mathscr{H}$ and the expected value of measuring an observable A for a system in a state ξ is $\langle A\xi, \xi\rangle$. Two observables A and B can be simultaneously measured with arbitrary precision only if they commute, i.e., $AB = BA$. The importance of this relation justifies the following notation.

DEFINITION. If X is a set of operators on $\mathscr{H}$, the commutant X' is the set of all *bounded* operators on $\mathscr{H}$ which commute with any element of X.

The time evolution of the system is governed by a privileged observable H and is given by the one-parameter unitary group $\exp(itH)$. Any symmetry group of the system will be implemented by a projective unitary representation of the group on H.

The above formalism was well understood and clearly enunciated by von Neumann in **[vN1]**. The whole picture can be almost deduced on more or less philosphical grounds from logical relationships between the results of experiments. See, for instance, **[Pi]**. One thing that is quite tricky from this "logical" point of view but which is an essential part of the picture is the description of a composite system made up of two or more subsystems. The formalism is that the Hilbert space describing the composite system is the tensor product of the Hilbert spaces describing the components. This is quite natural from the point of view of wave functions.

I have already said enough to justify, from a physical point of view, the study of von Neumann algebras, for which I now give three equivalent

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46L10, 81P05, 81R10, 57M25, 16S99.

0-8218-0167-8 \$1.00 + \$.25 per page

definitions, all on a given Hilbert space $\mathscr{H}$.

DEFINITION 1. A von Neumann algebra is the commutant of some selfadjoint set of operators on $\mathscr{H}$.

DEFINITION 2. A von Neumann algebra M is a selfadjoint algebra of operators on $\mathscr{H}$ equal to the commutant of its commutant: $M = M''$.

DEFINITION 3. A von Neumann algebra M is a selfadjoint algebra of bounded operators on $\mathscr{H}$, containing the identity and such that if a net A_n, in M, converges to A in the sense that $\langle A_n\xi, \eta\rangle \to \langle A\xi, \eta\rangle$ for all ξ and η, then $A \in M$ (the net is said to converge weakly).

Definitions 1 and 2 are superficially equivalent while the equivalence between 2 and 3 is the famous "density theorem" of von Neumann [**vN2**].

We can immediately see four (related) uses for von Neumann algebras.

(1) In the mathematical formulation of quantum mechanics (e.g., subsystems).
(2) To study unbounded operators by bounded ones using the commutant.
(3) To study unitary group representations.
(4) As a generalization of finite dimensional semisimple algebras where the selfadjointness of the algebra should guarantee some kind of semisimplicity.

All the above were stated motivations in the 1936 paper of Murray and von Neumann [**MvN**], where von Neumann algebras were first introduced under the name "rings of operators."

Abelian von Neumann algebras are easy to understand. Take a σ-finite measure space (X, μ) and form the Hilbert space $L^2(X, \mu)$. The L^∞ functions act on L^2 by multiplication and form a maximal abelian $*$-algebra of bounded operators. Thus $L^\infty(X, \mu)$ is an abelian von Neumann algebra. The general abelian von Neumann algebra can be obtained from this one simply by changing the "multiplicity," for instance by letting it act diagonally on $L^2(X, \mu) \oplus L^2(X, \mu) \oplus \cdots$, and then reducing to some invariant subspace. In any case, as an abstract $*$-algebra any abelian von Neumann algebra is isomorphic to $L^\infty(X, \mu)$.

Now if we take an arbitrary von Neumann algebra M, its *center* $Z(M)$ is abelian. Along with the decomposition of $Z(M)$ as $L^\infty(X, \mu)$ goes a decomposition of all of M as a "direct integral" $M = \int_X^\oplus M(\eta)d\mu(\eta)$ into a measurable family $M(\eta)$ of von Neumann algebras with trivial center. An arbitrary element of M becomes a measurable function $\eta \mapsto a(\eta) \in M(\eta)$, modulo sets of measure zero. As one can imagine, the details of the direct integral decomposition are somewhat painful (see [**vN3**]) but in principle it reduces the study of arbitrary von Neumann algebras to that of ones with trivial center. Note also how we are quickly moving away from the notion of a concrete von Neumann algebra to an abstract notion.

So what does a von Neumann algebra M with trivial center look like? In finite dimensions the structure theory of semisimple algebras says that such an algebra is $*$-isomorphic to a full matrix algebra $M = M_n(\mathbb{C})$. Moreover, as a concrete algebra on a Hilbert space it corresponds to a tensor product factorization $\mathscr{H} = \mathscr{H}_1 \otimes \mathscr{H}_2$ with $M = \{x \otimes 1 | x \text{ on } \mathscr{H}_1\}$ (and $M' = \{1 \otimes y | y \text{ on } \mathscr{H}_2\}$). The same factorization result holds in infinite dimensions if M is abstractly isomorphic to the algebra $\mathscr{B}(\mathscr{K})$ of all bounded operators on some Hilbert space. Thus Murray and von Neumann called a von Neumann algebra with trivial center a *factor*, and quickly restricted their attention to these.

In the language of quantum mechanics, a factor corresponds to a physical system for which there is no observable which can be simultaneously measured, to arbitrary precision, with all other observables. In group representation language, a representation whose commutant is a factor is called isotypical, at least in finite dimensions, because Schur's Lemma guarantees that only one type of representation occurs in a decomposition into irreducibles.

Central to the analysis of von Neumann algebras is the understanding of *projections* onto closed subspaces which can be characterized abstractly as operators p with $p^2 = p = p^*$. It is in terms of projections that Murray and von Neumann gave their first classification of factors into types I, II, and III. I shall give an alternative version emphasizing *traces*, though even so the role of projections will be important.

A factor M is said to be of *type* I if it is abstractly isomorphic to some $\mathscr{B}(\mathscr{K})$. It then admits a trace defined on a certain subalgebra of operators, such that the trace of a rank one projection is 1. With an obvious convention concerning ∞, type I factors (on a separable Hilbert space) are classified up to abstract isomorphism by the single number $n = \dim \mathscr{K}$, and up to concrete isomorphism by the pair $(\dim \mathscr{K}, \dim \mathscr{K}')$, where $\mathscr{H} = \mathscr{K} \otimes \mathscr{K}'$ as before. The traces of projections form the set I_n: $\{0, 1, 2, \ldots, n\}$.

A factor M is said to be of *type* II_1 if it is not of type I and admits a trace tr which is a linear functional $\mathrm{tr}: M \to \mathbb{C}$ satisfying

(1) $\mathrm{tr}(ab) = \mathrm{tr}(ba)$,
(2) $\mathrm{tr}(1) = 1$,
(3) $\mathrm{tr}(aa^*) > 0$ if $a \neq 0$,
(4) tr is weakly continuous.

(As it happens, only condition (1) is important, (2) is a normalization, and (3) and (4) are automatic.)

The traces of projections in a II_1 factor form the set $[0, 1]$ and one talks of continuous dimensionality.

It is not true that II_1 factors give tensor product factorizations, nor is it known how to list all II_1 factors up to abstract isomorphism. However Murray and von Neumann did show how to associate a number $\dim_M(\mathscr{H})$ (which they called the coupling constant) to a concrete II_1 factor on $\mathscr{H}$, which completely characterizes M, once it is known as an abstract II_1

factor. It corresponds to the integer $\dim \mathscr{H}'$ in the type I case but because of continuous dimensionality it may be any nonnegative real or ∞.

A factor M is said to be of *type* II_∞ if it contains a projection p with pMp a type II_1 factor. It is then of the form $N \otimes \mathscr{B}(\mathscr{H})$ with $N \cong pMp$, and admits a trace in a similar sense to the type I case. The traces of projections form the set $[0, \infty]$.

A factor M is said to be of *type* III if it is not of type I or II.

It is easy to construct II_1 factors (and hence II_∞) using discrete groups. If Γ is a group all of whose conjugacy classes (except that of the identity) are infinite, then the von Neumann algebra generated by the left regular representation is a type II_1, factor. Elements are all of the form $\sum_{\gamma\in\Gamma} c_\gamma U_\gamma$ (where $(u_\gamma f)(\gamma^1) = f(\gamma^{-1}\gamma^1)$, $f \in l^2(\Gamma, \mathbb{C})$) and $\mathrm{tr}(\sum c_\gamma u_\gamma) = C_1$. Type III factors can be constructed by replacing the scalars in the above construction by an abelian von Neumann algebra A carrying an ergodic action of Γ with no invariant measure (on the space (X, μ) for which $A = L^\infty(X, \mu)$). In quantum field theory, the von Neumann algebra corresponding to all fields restricted to a bounded region of space time is supposed to be a type III factor and this can be shown in some cases.

To begin to answer more subtle questions, such as "how does the II_1 factor depend on the group Γ as above," the notion of *hyperfiniteness* is crucial. A von Neumann algebra M is said to be hyperfinite if there is an increasing net of finite-dimensional $*$-subalgebras whose union is weakly dense in M. Murray and von Neumann showed in **[MvN]** that there is only one hyperfinite II_1 factor. A celebrated result of Connes in **[Co1]** asserts that the factors constructed above using groups are hyperfinite iff Γ is amenable (e.g., $\Gamma = \mathbb{Z}$). The local algebras of quantum field theory are hyperfinite.

The classification of hyperfinite factors is complete. To explain it would require some Tomita-Takesaki theory so we simply say that Connes gave a finer classification of type III factors in terms of a continuous parameter $\lambda \in [0, 1]$. Connes showed that there is exactly one hyperfinite type II_∞ factor and one in each class III_λ, $0 < \lambda < 1$. By results of Krieger, hyperfinite type III_0 factors are classified by ergodic transformations, and Haagerup proved the uniqueness of the hyperfinite type III_1 factor in **[Ha]**.

Subfactors. With hyperfinite factors classified it is natural to turn to subfactors. There were four good reasons for this in 1976 when I began to look at subfactors.

(1) The extraordinary result of Connes in **[Co1]** that any subfactor of the hyperfinite II_1 factor R is itself hyperfinite, hence isomorphic to R or finite dimensional.

(2) The growing conviction, expounded in **[Co2]**, that II_1 factors could be thought of as scalars, and Hilbert spaces carrying representations of them as "vector spaces" over the II_1 factors. Hence the notation $\dim_M(\mathscr{H})$ for the Murray-von Neumann coupling constant. By this analogy, the study of

subfactors corresponds to the rich subject of Galois theory.

(3) A nontrivial but barely noticed result of Goldman **[Go]** which is quite analogous to the fact that a subgroup of index 2 of a group is normal.

(4) Remarkable success, beginning with **[Co3]**, in classifying group actions on the hyperfinite II_1 factor.

The following invariant of a subfactor (up to conjugacy by automorphisms) was implicit in works mentioned above and was probably thought about by Murray and von Neumann.

DEFINITION. If N is a subfactor of the II_1 factor M containing the same identity and if M acts on $\mathscr{H}$ so that M' is a II_1 factor, the real number $\frac{\dim_N(\mathscr{H})}{\dim_M(\mathscr{H})}$ is independent of $\mathscr{H}$ and is called the *index of N in M* or the *degree of the extension M of N*. It is written $[M: N]$ and trivally $[M: N] \geq 1$.

In ring-theoretic terms $[M: N]$ measures the rank of M as a left N-module. Since $K_0(N) = \mathbb{R}$, this is a precise statement.

The simplest example is where $M = N \otimes M_n(\mathbb{C})$, N being identified with $N \otimes 1$. Here $[M: N] = n^2$.

The next example is where $N = M^G$, the fixed point algebra for a group of automorphisms. Provided every element of G is outer (except the identity), we have $[M: N] = |G|$.

There are two extremes for subfactors, *irreducible* and *locally trivial*. An irreducible subfactor is one for which $N' \cap M = \mathbb{C}1$, and a locally trivial subfactor is one for which $[pMp: pNp] = 1$ for all minimal projections in $N' \cap M$. The usefulness of the locally trivial subfactors has only recently become apparent in work of Popa (**[Pop]**).

The subfactors $N = M^G$ as above are irreducible and show that the numbers 1, 2, and 3 are indices of subfactors. If a subfactor is locally trivial it is easy to see that its index, if different from 1, must be ≥ 4. An obvious construction of locally trivial subfactors shows that the hyperfinite II_1 factor has subfactors of index r for any $r \geq 4$. Thus one is tempted to ask, "what are the values of the index for subfactors?" The result, proved in **[JO1]** was a surprise. If $[M: N] < 4$ then it must be of the form $4\cos^2 \pi/n$ for some integer $n \geq 3$, and all such values are realized for subfactors of the hyperfinite type II_1 factor.

The situation for indices of irreducible subfactors is unclear. It seems plausible that there is a gap between 4 and the next value.

Up to now all of our results have been in the *internal* theory of von Neumann algebras. Historically there have been many connections with other subjects. We have already alluded to the connection with foliations in **[Co2]**. We are about to pursue what has been an extremely faithful connection with several subjects. This came from a rather unlikely source—a detail in the proof of the result on index values of subfactors. So we must begin that proof.

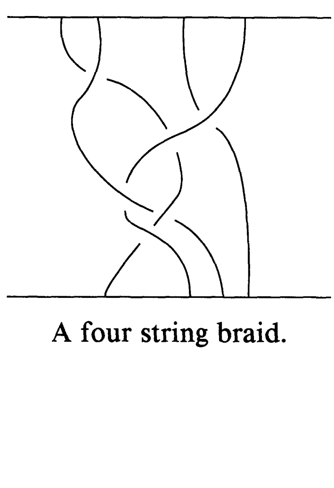
Starting with a subfactor $N \subseteq M$, a general result of [U] asserts that there is a "conditional expectation" $e_N \colon M \to N$ which is an $N-N$ bimodule map preserving 1. It follows that if $[M\colon N] < \infty$, the algebra of linear maps of M generated by e_N and M itself (acting by left multiplication) is again a II_1 factor, with $\mathrm{tr}(e_N) = [M\colon N]^{-1}$. Let us call this canonical construction $\langle M, e_N \rangle$. By results of [MvN] we have $[\langle M, e_N\rangle \colon N]$. One may now continue this process by considering $M \subset \langle M, e_N \rangle$ and so on. One obtains a tower M_i of II_1 factors defined by the second-order difference equation $M_{i+1} = \langle M_i, e_{M_{i-1}} \rangle$ with initial conditions $M_1 = N$, $M_2 = M$. Letting $e_i = e_{M_i}$ and noting that the trace tr is unique on a II_1 factor, and hence well defined on $\bigcup_i M_i$, we have obtained, among other things, a sequence of operators e_i satisfying

(i) $e_i^2 = e_i^* = e_i$,
(ii) $e_i e_{i\pm 1} e_i = \tau e_i$ $(\tau = [M\colon N]^{-1})$,
(iii) $e_i e_j = e_j e_i$ if $|i-j| \geq 2$,
(iv) $\mathrm{tr}(x e_{n+1}) = \tau\,\mathrm{tr}(x)$ if x is a word on $1, e_1, e_2, \ldots, e_n$.

The last property is called the Markov property because those e_i's can be thought of as defining a "noncommutative Markov chain." The proof of the result on restrictions of the index is completed by examining the positivity condition $\mathrm{tr}(a^* a) > 0$ for $a \neq 0$ for the algebra generated by the e_i's.

Precisely the relations (i), (ii), and (iii) had been noticed and used by Temperley and Lieb in statistical mechanics in [TL], but this connection was not noticed at once. It will become highly significant later on. But the first connection to be observed (by colleagues in Geneva) was a similarity between (ii) and (iii) and Artin's presentation of the braid group. Thus I need to explain about the braid group.

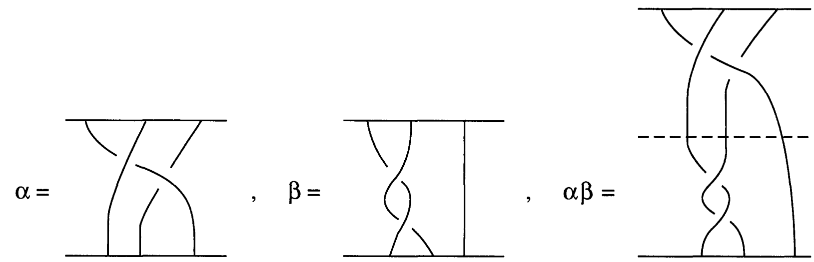
Braids, knots, and links. An n-string *braid* is a way of tying n points on a horizontal plane to the same points on another horizontal plane so that the height function on any string has no critical points. It is conventional, and apparently quite important, to arrange the n points on a straight line and to number them 1 to n in increasing order. Thus one may draw any braid as shown below.



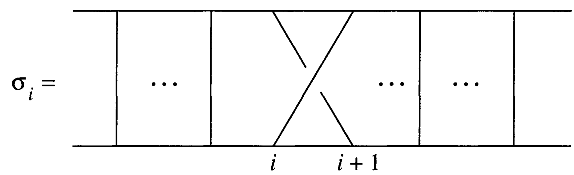
A four string braid.

Braids are considered up to an intuitive equivalence relation which may be rigorously defined using homotopy language.

The significant thing about braids on n strings is that they form a *group* B_n under concatenation as illustrated below:



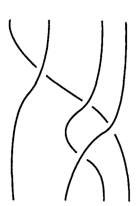
The following elementary braids generate the braid group:



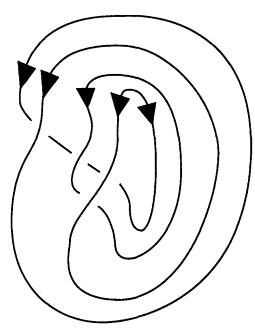
Artin showed in **[Ar]** that the following is a presentation of B_n:

$$\langle \sigma_1, \sigma_2, \dots, \sigma_{n-1} | \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \ \sigma_i\sigma_j = \sigma_j\sigma_i \text{ for } |i-j| \geq 2\rangle.$$

A knot in S^3 is a smoothly embedded circle and a link is a nonintersecting union of knots. All are to be considered up to diffeomorphisms of S^3. Given a braid $\alpha \in B_n$, one may obtain an oriented link $\hat{\alpha}$ in S^3 as depicted below.



braid $\alpha \in B_4$



oriented link $\hat{\alpha}$

In the light of events since 1984, this process should be called taking the trace of the braid and the result as some kind of link valued character of the braid. But it is traditional (see **[Bi]**) to call $\hat{\alpha}$ the closure of the braid and we shall use this terminology.

A result of Alexander **[Al]** asserts that any oriented link can be obtained in this way and Markov (son of the Markov of Markov chains!) showed that two braids $x \in B_n$ and $\beta \in B_m$ have the same closures if they can be connected by a sequence of "Markov moves" as follows:

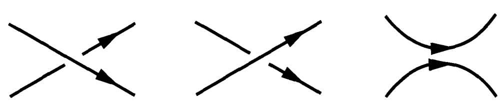
type I *Markov move*: $\alpha \to \gamma\alpha\gamma^{-1}$, $\alpha \in B_n$, $\gamma \in B_n$,

type II *Markov move*: $\alpha \leftrightarrow \alpha\sigma_n^{\pm 1}$, $\alpha \in B_n$, and B_n is embedded in B_{n+1} according to the points on the straight line.

The similarity between this business and relations (i)–(iv) in the algebra coming from subfactors gave, after much conversation between the author and topologists, especially Joan Birman, an invariant of oriented links in S^3, called $V_L(t)$, as follows (see **[Jo2]**).

Define a representation π of B_n, for all n, inside the algebra generated by the e_i's coming from the subfactor proof by $\pi(\sigma_i) = t^{1/4}(te_i - (1-e_i))$. If L is a link choose $\alpha \in B_m$ with $\hat{\alpha} = L$. Then $V_L(t) = (-\sqrt{t} - \frac{1}{\sqrt{t}})^{m-1}\,\mathrm{tr}(\pi(\alpha))$ depends only on L by Markov's result.

Property (i), $e_i^2 = e_i$, is easily seen to imply that if L_+, L_-, and L_0 are links with diagrams differing only at one crossing as below,



L_+ $\qquad$ L_- $\qquad$ L_0

then $\frac{1}{t}V_{L_+} - tV_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})V_{L_0}$. The Alexander polynomial $\triangle_L$ satisfies $\triangle_{L_+} - \triangle_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}})\triangle_{L_0}$. This relation, together with the normalization $V_{\bigcirc} \equiv 1$, $\triangle_{\bigcirc} \equiv 1$ suffices to calculate these polynomials. One finds

$$V_{\text{trefoil}} = t + t^3 - t^4, \qquad \triangle_{\text{trefoil}} = t^{-1} - 1 + t.$$

Immediately after its discovery V was generalized in **[F+]** to a two-variable version $P(l, m)$, embracing the Alexander polynomial, with defining formula $l^{-1}P_{L_+} - lP_{L_-} = mP_{L_0}$. This polynomial P contains significant information about turning links into braids **[Mo]**, **[FW]**.

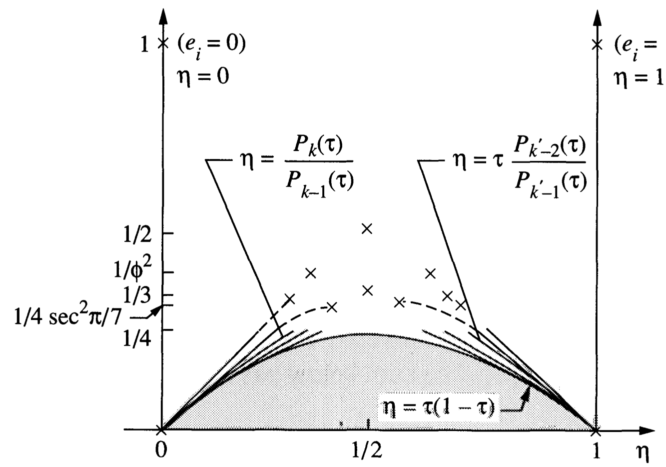
Ocneanu's approach to defining P_L was important for von Neumann algebras for he introduced the Hecke algebra of type A_n with presentation $\langle g_1, g_2, \dots, g_{n-1} | g_i^2 = (q-1)g_i + q,\ g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1},\ g_i g_j = g_j g_i$ for

$|i-j| \geq 2\rangle$ and defined P via a trace on this algebra as we did above with the Temperly-Lieb algebra (which is a Hecke algebra quotient). The trace is defined like condition (iv) above:

$$\operatorname{tr}(w g_n) = z \operatorname{tr}(w)$$

where w is a word on $g_1, g_2, \ldots, g_{n-1}$ and z is a free parameter. Ocneanu also showed that the Hecke algebra for A_∞, together with this trace, defines a II_1 factor precisely for the following values.

$$\tau^{-1} = 2 + q + q^{-1}, \eta =$$



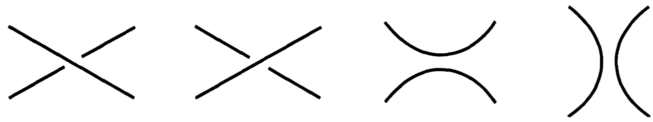
One can define a subfactor by taking the algebra generated by $g_2, g_3, g_4, \ldots$. Wenzl **[We1]** calculated the index for $\tau^{-1} = 4\cos 2\pi/n$, η to be $\frac{\sin^2 k\pi/n}{\sin^2 \pi/n}$. He also proved that these subfactors are irreducible.

The most spectacular success of the new polynomials has been the proof of some Tait conjectures (**[K2]**, **[Mus]**, **[Th]**). We have the following remarkable inequality, true for a link diagram L with n crossings:

(a) $\deg(V_L(t)) \leq n$,
(b) $\deg(V_L(t)) = n$ iff L is *alternating* and *reduced*.

Here "deg" means, for instance, $\deg(t + t^3 - t^4) = 4 - 1 = 3$. Alternating means that over-crossings and under-crossings alternate as any component of the link is followed and "reduced" means that the diagram admits no obvious simplification. It follows immediately from (a) and (b) that if the link L has a reduced alternating diagram with n crossings then it has no diagram at all with fewer crossings. For an account of this see **[HKW]**.

The Kauffman polynomial. Following the discovery of V and P, a polynomial of unoriented links Q_L was found in **[BLM, Ho]**. It is defined by the relation $Q_{L_+} + Q_{L_-} = x(Q_{L_0} + Q_{L_\infty})$, where L_+, L_-, L_0, and L_∞ are



L_+ L_- L_0 L_∞

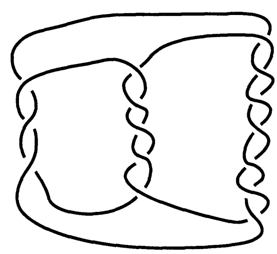
as shown: (and $Q_{\bigcirc} \equiv 1$). A sample calculation gives $Q = 2x^2 + 2x - 3$. Kauffman saw how to give a two-variable version of Q by following the inductive scheme for calculating Q, but whenever a picture like is encountered it is to be replaced by ; the corresponding polynomial must be multiplied by a factor "a" (or a^{-1} for). If the original link L is then given an orientation, the polynomial obtained by this computation may be multiplied by a power of "a" to give the Kauffman polynomial $F_L(a, x)$. The Kauffman polynomial contains V_L as a specialization but not Δ. See **[K2]**.

There are many open questions about these polynomials. The most obvious are:

(I) Is there a nontrivial knot K with $V_K(t) \equiv 1$? (Same for P, Q, F).

(II) Can one characterize which Laurent polynomials arise as $V_K(t)$ for some knot? (Same for P, Q, F).

Both these questions were answered for the Alexander polynomial in a paper by Seifert in 1934 [S]. The knot below has trivial Alexander polynomial.



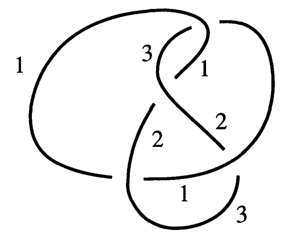
A Laurent polynomial $p(t)$ is $\Delta_K(t)$ for some K iff $p(1) = 1$ and $p(t) = p(t^{-1})$. Seifert solved these problems by understanding the cyclic covering spaces of the knot complement in a very concrete way using an oriented surface whose boundary is the knot.

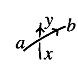
These problems seem out of reach for V at the moment. Barring an accidental discovery, a new idea is needed in the understanding of these polynomials. Everything seems to indicate that the place to look is physics, as I shall now explain.

Statistical mechanical models. We now begin to return to physics. In classical statistical mechanics a system may be defined by its set of *states* and a function $E(\sigma)$ which assigns to a state σ its energy. The *partition func-*

tion of the system is then defined to be $Z = \sum_{\text{states }\sigma} \exp\left(-\frac{E(\sigma)}{kT}\right)$. One is typically interested in very large but regular systems (e.g., on a square lattice) so the calculation of Z often proceeds by introducing some algebra of matrices. This is the context in which Temperley and Lieb discovered the interest of relations (i), (ii), and (iii) for the e_i's. They used these relations to show the equivalence between two models (self-dual Potts and 6-vertex). In this algebra the partition function will typically be the trace of some matrix representing the system. This suggests looking for a statistical mechanical model to give the polynomials we have discussed. This suggestion was totally compelling in the light of an explicit formula for V_L given by Kauffman **[K1]**. Indeed a glance at Chapter 12 of **[Ba]** shows that Kauffman's "states model" is precisely an intermediate step in an elegant combinatorial proof of Temperley-Lieb equivalence.

Thus one is led to define vertex models and spin models directly on link diagrams (see **[Jo3]**). The vertex model scheme is as follows. We begin with two sets of "Boltzmann weights" $w_{\pm}(a, b|x, y)$, where a, b, x, y are "spin" indices, say running from 1 to N. If L is a link diagram we define a state of it to be a function from the edges of the diagram to $\{1, 2, \ldots, N\}$. A state of the figure eight is depicted below:



Given a state every crossing is surrounded by a configuration 

. One then takes the product $\prod_{\text{crossings}} w_{\pm}(a, b|x, y)$, using the " + " weights at a positive crossing and the " − " weights at a negative one. The partition function Z_L is then defined to be $Z_L = \sum_{\text{states}} \prod_{\text{crossings}} w_{\pm}(a, b|x, y)$. (In fact another term is needed to make the following discussion correct but I shall ignore this for simplicity. See **[Tu, Re]** or **[Jo3]**.)

One may now inquire as to whether $w_{\pm}(a, b|x, y)$ may be chosen in such a way that Z depends only on the link and not on the chosen diagram. The answer is yes and an abundance of examples may be constructed using the theory of quantum groups. Indeed the index sets $\{1, \ldots, N\}$ may be chosen differently for different components of the link. The final result, proved in **[Ro, Re]** is as follows: let $\mathfrak{G}$ be a simple complex Lie algebra and let L be a link with distinguished components $c_1, c_2, \ldots, c_n$. Then to every way of assigning finite dimensional representations $\pi_1, \ldots, \pi_n$ to $c_1, \ldots, c_n$ there is an invariant $Z_L^{\mathfrak{G}}(\pi_1, \pi_2, \ldots, \pi_n, q)$ defined as the partition function for some choice of $w_{\pm}(a, b|x, y)$.

Quantum groups were actually constructed with a view to defining and understanding statistical mechanical models for which the large scale behavior of the partition function can be calculated explicitly. A condition that seems to guarantee this has been emphasized by Baxter and is known as the Yang-Baxter equation. The reason that quantum groups apply to knot theory is that the main condition on the Boltzmann weights that ensures topological invariance of the partition function is a weak form of the Yang-Baxter equation. For references on quantum groups see **[Dr, Ji1]**.

Is this appearance of the statistical mechanical formalism a coincidence? At this stage one cannot give a definite answer to this question but one may begin by seeing if the correspondence can be pushed further. Vertex models are not the only kind used in statistical mechanics. There are also "spin" models, where the spins live on the vertices of a graph, and IRF models, where the energy of a state is the sum of energy contributions from the faces of a graph. Both of these kinds of models adapt beautifully to knot theory. There are two simple spin models which give the V polynomial and the homology of the 2-fold branched cover of S^3 branched over the link respectively. See **[Jo3]**. The IRF models give what is probably the most economical model (in terms of number of states) known. A beautiful account occurs in **[KR]**.

I have not yet explained how the invariants coming from quantum groups relate to the polynomials V, P, Q, and F. The simplest case is $V_L(t)$ which is $Z_L^{\mathrm{sl}_2}(\pi_2, \pi_2, \dots, \pi_2, t)$, π being the 2-dimensional defining representations. Also $Z_L^{\mathrm{sl}_n}(\pi_n, \pi_n, \dots, \pi_n)$ is $P_L(q^{(n+1)/2}, q^{1/2} - q^{-1/2})$ and the orthogonal and symplectic algebras, with the defining representation assigned to all components of the link, give an infinite sequence of specializations of the Kauffman polynomial.

Note that if the knot diagram is a closed braid, the partition function Z is easily seen to be the trace of the braid in a representation of the braid group coming from the $w_\pm(a, b|x, y)$ on $\bigotimes^n V$, where σ_i acts on $\bigotimes^n V$ by

$$\begin{aligned}\sigma_i(v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_n) \\ = \sum w(i, j|i+1, j+1)(v_1 \otimes \cdots \otimes v_j \otimes v_{j+1} \otimes \cdots \otimes v_n).\end{aligned}$$

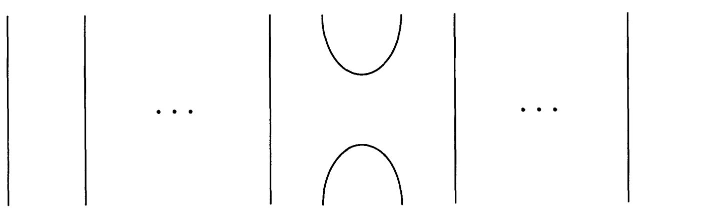
This brings us back to the braid way of calculating the polynomials. In fact one has to modify the trace a little for this to work. One obtains "states" on $\bigotimes_{n=1}^{\infty} \mathrm{End}(V)$ which give rise naturally to type III_λ factors, $0 < \lambda < 1$ (see **[Pow]**).

Quantum invariant theory. A central problem in classical invariant theory is to decompose the tensor powers of a given irreducible representation of a group. This is of course equivalent to finding the commutant algebras. The simplest case is for sl_n in its defining representation V. The symmetric group S_m acts naturally on $\bigotimes^m V$ and it is well known that the algebra generated by S_m is the commutant of the tensor product action of sl_n. Thus the group algebra of S_m is a universal object for the commutants. For groups other than SL_n it is necessary to find other invariants to generate the whole commutant. Brauer gave a universal algebra for the commutants of the orthogonal and symplectic groups in **[Br]**. An old problem about this algebra was solved by Wenzl using the techniques described in this talk—see **[We2]**.

If we replaced $\mathfrak{G}$ by the quantum group $U_h(\mathfrak{G})$ one may ask the same question about the commutant. Jimbo proved in **[Ji2]** that the commutant of $U_h(\mathrm{sl}_n)$ on $\bigotimes^m V$ is generated by the braid group representation coming from the representation on $\bigotimes^m V$ we have just described! Thus the P polynomial sits in a sense in the commutant of sl_n.

The situation for the orthogonal and symplectic algebras is even more interesting. We will see that the Kauffman polynomial plays exactly the same role as the P polynomial did for sl_n. In **[BW, Mus]**, an algebra, the BMW algebra, was invented to play the role for the Kauffman polynomial that the Hecke algebra plays for P. The idea, due to Kauffman, was to extend the braid group by allowing objects

$E_i =$



as well as σ_i's in braid words. To mimic the Kauffman polynomial definition, an algebra with generators G_i, E_i, $i = 1, \ldots, n-1$ was defined with relations

$$
\begin{aligned}
&G_iG_{i+1}G_i = G_{i+1}G_iG_{i+1}, \quad G_iG_j = G_jG_i \quad \text{if } |i-j| \geq 2,\\
&G_i + G_i^{-1} = x(1 + E_i),\\
&E_iG_i = G_iE_i = aE_i,\\
&E_i^2 = (a + a^{-1} - x)x^{-1}E_i,\\
&E_iG_{i\pm1}^{\pm1}E_i = a^{\mp1}E_i,\\
&E_iG_{i\pm1}G_i = E_iE_{i\pm1}.
\end{aligned}
$$

These relations can be interpreted by drawing pictures. A trace was defined on this algebra using the Kauffman polynomial, with the property that the appropriately normalized trace of a braid is the Kauffman polynomial of the closure of the braid.

By results of Jimbo in **[Ji1]**, the braid group representations coming from the quantum versions of the orthogonal and symplectic groups in their defining representations, factor through the BMW algebra in the obvious way. It is surely true that they generate the commutants of the quantum group representations, though we have not seen a complete proof of this.

It seems important to decide in general how much of the commutant is generated by the braid matrices. For sl_n, they generate the whole commutant in all irreducible representations (see **[KR]**) but this fails for the adjoint representation of sl_3. On the other hand, if one includes the whole family $R(\lambda)$, λ being the spectral parameter, one will obtain more of the commutant. The significance of this question is that all the classical invariants might appear simply as limits of a very simple picture on the quantum level.

Quantum field theory. One reason for the interest in statistical mechanical models and their phase transitions is that at the critical point they are supposed to give nontrivial examples of continuum quantum field theories by letting the lattice spacing tend to zero. In two dimensions the resulting field theory should carry a projective unitary representation of the conformal group. The corresponding (complexified) Lie algebra is the direct sum of two copies of the "Virasoro algebra" with basis $\{c, L_n; n \in \mathbb{Z}\}$ with Lie bracket $[L_n, L_m] = (n-m)L_{n+m} + (\frac{m^3-m}{12})\delta_{n,-m}c$, $[c, L_n] = 0$ for all n. In **[FQS]** it was shown that if $c < 1$ in a unitary ($L_n^* = L_{-n}$) irreducible representation of the Virasoro algebra then $c = 1 - \frac{b}{n(n+1)}$. This result is analogous to the result for indices of subfactors, less than 4. This, and other ideas, suggest that the way to make a direct connection between knot theory and von Neumann algebras is via quantum field theory.

Witten, **[Wi]** has given a definition of the knot polynomials as partition functions for three-dimensional quantum field theories with Chern-Simons action in a way intimately related with the Wess-Zumino-Witten (conformal) quantum field theories. A great advantage of this approach is that it works for links in any oriented three-manifold, and, for the empty link, even gives an invariant of considerable interest. See Witten's lecture in this series.

In a rather different approach, Fredenhagen, Longo, Rehren, and Schroer **[FRS**, L] have discovered the braiding structure and indices for subfactors to be consequences of an axiomatic approach to "superselection sectors" in algebraic quantum field theory. The equivalence, beyond some bounded region, of a representation of the field algebra with the vacuum representation gives an endomorphism of the local field algebra, and the braiding operators come from the representation of the Lorentz group. Fröhlich has had ideas along similar lines, see **[F]**.

These connections with quantum field theory are currently under active study and it is hoped that a beautiful unified theory will appear.

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