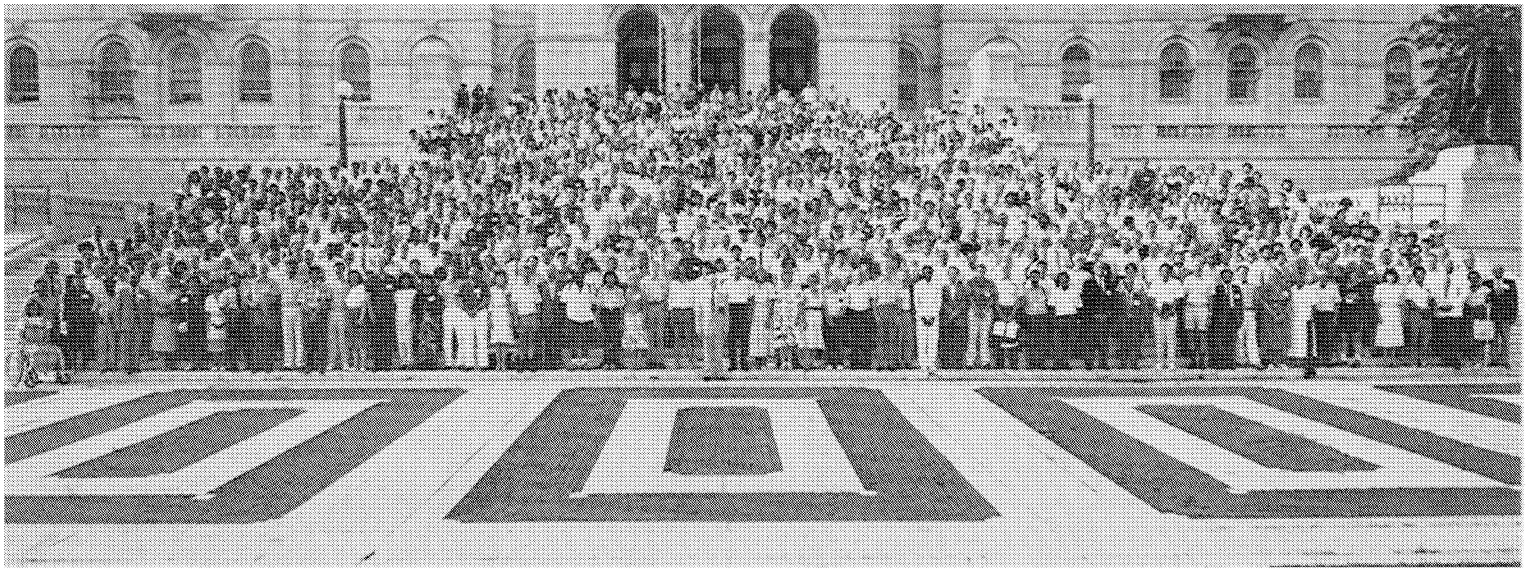
Mathematics into the Twenty-first Century

1988 Centennial Symposium
August 8–12



Statehouse Reception

AMERICAN MATHEMATICAL SOCIETY
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Volume II

Mathematics into the Twenty-first Century

1988 Centennial Symposium
August 8–12

Felix E. Browder
Editor

American Mathematical Society
Providence, Rhode Island
1992

MATHEMATICS INTO THE TWENTY-FIRST CENTURY
1988 CENTENNIAL SYMPOSIUM
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Contents

Preface

This volume contains the written versions of sixteen of the original eighteen addresses presented at the American Mathematical Society's Centennial Symposium *Mathematics into the Twenty-first Century* held from August 8–12, 1988. These talks, delivered at the Providence Performing Arts Center, were the principal component of the Scientific Program at the Centennial Celebration. Attendance at this meeting was unprecedented for AMS Summer Meetings with 1,949 members of the Society and a total of 2,502 in attendance including spouses, guests, etc.

The Centennial Celebration was organized by the Centennial Committee with the following members: Felix E. Browder, Rutgers University; Harold M. Edwards, Courant Institute of Mathematical Sciences, New York University; Andrew M. Gleason, Harvard University, a former President of the American Mathematical Society; George Daniel Mostow, Yale University, then current President of the American Mathematical Society; and Everett Pitcher, Chairman, Lehigh University.

The Symposium was organized by the Centennial Program Committee whose members were: Hyman Bass, Columbia University; Felix E. Browder, Chairman; Phillip A. Griffiths, Duke University; John W. Milnor, Institute for Advanced Study; Cathleen S. Morawetz, Courant Institute of Mathematical Sciences, New York University.

Dr. Edward E. David, Jr., Keynote Speaker, delivered the general address entitled *Renewing U.S. Mathematics: an Agenda to Begin the Second Century.* By invitation of the AMS–MAA Joint Program Committee, three retrospective talks were given by Raoul Bott, Peter Lax, and Saunders Mac Lane. The written versions of these talks have been published in *A Century of Mathematics in America*, Parts II and III.

The Centennial Committee thanks the National Science Foundation for its support of the symposium *Mathematics into the Twenty-first Century* (Grant #DMS8716887), and the Department of Energy (Grant #DE-FG02-88ER25056), the Office of Naval Research (Grant #N00014-88-J-1096), and the United States Army Research Office (Grant #DAAL03-88-G-0022) for grants supporting travel and subsistence for young mathematicians attending the Centennial.

Introduction

The 1988 Symposium *Mathematics into the Twenty-first Century*, of which the present volume is the Proceedings, was organized to celebrate the hundredth anniversary of the founding of the American Mathematical Society (AMS). It was developed on a set of principles which differed in a number of respects from other commemorative celebrations, in particular from the 1938 Semi-centennial Celebration of the AMS. Though the Centennial gave rise to a series of three historical volumes, the Symposium was not historical in character nor did it itself contain speeches of reminiscence or celebration. Three talks of this sort (by Bott, Lax, and Mac Lane) were indeed delivered during the week of the Symposium, but the speakers and their commission were chosen by an entirely separate AMS-MAA organizing committee.

The main principles on which the Symposium was organized can be summarized as follows:

I. The talks should cover as many as possible of the most important central directions of contemporary mathematical research.

II. As far as we could choose, the speakers should be individuals of stature in these directions who have done their principal work in the United States and who are likely to be principal contributors after the year 2000.

III. The central topics of the talks should include not only pure mathematics in its classical forms but also the development of the rapidly developing interaction of sophisticated mathematics with front line areas in science and engineering—in physics, fluid dynamics, computational science, biology, statistics, and computer science.

IV. The talks ought to have an expository intent to make it possible for a broad audience of mathematicians and mathematical practitioners to understand as much as possible of the spirit of what mathematics has accomplished in the last fifty years and of what it hopes to accomplish in the next fifty years.

The Symposium took place with eighteen speakers (culled from an original list of twenty-four) and in the judgment of most people to whom we have spoken, it was an overwhelming success. In the middle of the summer in Providence, 2000 mathematicians came and actually listened to the talks. The speakers tried to be understood and often succeeded. Morale was high, indeed exceptionally so, and the intent of the celebration was vigorously fulfilled.

What we can say of the present volume of Proceedings in terms of the principles outlined above?

Sixteen of the eighteen speakers contributed to the Proceedings, albeit with a lot of coaxing in the process. In the case of Thurston and Tarjan, the two who did not, we got what we would reasonably expect. Thurston is almost a mythical figure in terms of his erratic record in publication, and Tarjan (as he remarked from the beginning) is clearly overcommitted by several hundred percent. There is one shortcoming in the volume that arises from Tarjan's defection: the lack of any contribution involving theoretical computer science and its combinatorial substructure.

The remaining fourteen speakers fulfilled their commitments. In one case, that of Roger Howe, some might say that by writing an "article" which is an expository book in itself on Lie theory and its applications, he has overfulfilled his commitment. My answer to objectors on grounds of uniformity, is: What is the harm? If every speaker had devoted as much effort and mental energy to writing up his talk, we might have used up enormously more paper for this volume (or rather volumes) with an even more useful result. Certainly the expository intent was well fulfilled by Howe's contribution.

The subjects treated in this volume are a reasonable selection of what should be in it. The speakers appeared in alphabetical order and are presented in that order followed by their general areas of study: Michael Asbacher, finite group theory; Luis Caffarelli, nonlinear elliptic partial differential equations; Persi Diaconis, statistics and group invariance; Charles Fefferman, analysis in mathematical physics; Michael Freedman, low-dimensional topology; Harvey Friedman, mathematical logic; Benedict Gross, algebraic number theory; Joseph Harris, algebraic geometry; Roger Howe, Lie theory; Vaughn Jones, knot theory and von Neumann algebras; Victor Kac, Kac-Moody algebras; Andrew Majda, computational fluid mechanics and nonlinear analysis; Charles Peskin, computational methods in biological models; Dennis Sullivan, dynamics and Riemann surfaces; Karen Uhlenbeck, differential geometry, nonlinear elliptic equations and gauge theory; Edward Witten, geometry and quantum field theory.

Aside from the regrettable omission of computer science, the papers presented here form a significant panorama of most of the most vital mathematics of the present epoch. In 1990 Jones and Witten, two of the speakers, received Fields medals at the International Congress of Mathematicians at Kyoto.

We look upon this volume as an obvious symbol of the central role of American mathematics on the world mathematical scene, including the many vital contributors to American mathematics who have come to the U.S. from other lands, especially since the 1930s. The intense vitality of mathematics to which we can all testify owes a great deal to this dynamic synergy between America and the rest of the mathematical world. The American Mathematical Society has been one of the principal agents of this synergy and is one of its most prominent symbols.

Symposium Speakers

Michael Aschbacher
Professor of Mathematics
California Institute of Technology
Ph.D., University of Wisconsin, 1969

2:00 p.m.
Monday, August 8

Representations of finite groups as permutation groups
The classification of the finite simple groups in 1981 changed the landscape of finite group theory and led to an increased effort to describe the structure and representations of the simple groups. Together with the classification, this effort has made possible unexpected applications of finite group theory in other branches of mathematics.
Introduced by Daniel Gorenstein.

Luis A. Caffarelli
Professor of Mathematics
Institute for Advanced Study
Ph.D., University of Buenos Aires, 1972

3:15 p.m.
Monday, August 8

The geometry of solutions to nonlinear problems
This talk will discuss geometric techniques to study the shape and regularity of solutions to nonlinear elliptic equations and their level surfaces.
Introduced by Louis Nirenberg.

Reprinted from the *Notices of the AMS*, July/August 1988, pages 833–836.

Persi Diaconis
Professor of Mathematics
Harvard University
Ph.D., Harvard University, 1974

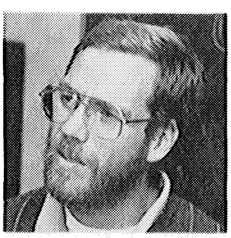
8:30 a.m.
Tuesday, August 9

Sufficiency as statistical symmetry
To judge what parts of a data set are worth saving, statisticians have developed a useful tool called *sufficiency*, which can be seen as an extension of the invariants of a group. Sufficiency allows a unified construction of statistical models, sheds light on the factorization of generating functions in combinatorics, and provides the underpinnings for recent work in statistical mechanics. This talk will explain the concept of sufficiency and survey these applications.
Introduced by Gian-Carlo Rota.

Charles L. Fefferman
Professor of Mathematics
Princeton University
Ph.D., Princeton University, 1969

9:45 a.m.
Tuesday, August 9

Problems from mathematical physics
This talk will cover two problems in mathematical physics. The first is from quantum mechanics and concerns the question of how large numbers of electrons combine with large numbers of protons to form large numbers of atoms. The second is from general relativity and concerns a proof that some small initial disturbance will not concentrate and become a black hole.
Introduced by Felix E. Browder.



Michael H. Freedman
Charles Lee Powell Chair Professor
University of California, San Diego
Ph.D., Princeton University, 1973

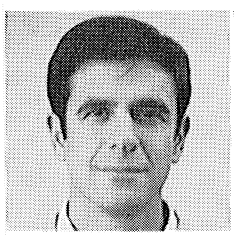
2:00 p.m.
Tuesday, August 9

Working and playing with the two-dimensional disk
The conformal structure of the disk is useful in studying the topology of (real) surfaces. A more combinatorial-topological study of maps of a disk has illuminated the study of three-dimensional manifolds. This talk will briefly survey the role of the disk in the theory of high-dimensional manifolds, and go on to address the special problems of a disk mapped into a four-dimensional manifold. This is the point at which the topological and smooth theories diverge, and some discussion of the disparities between them will be given.
Introduced by William Browder, President-Elect of the AMS.

Harvey M. Friedman
Professor of Mathematics
Ohio State University
Ph.D., Massachusetts Institute of Technology, 1967

3:15 p.m.
Tuesday, August 9

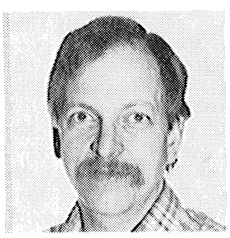
The incompleteness phenomena
By 1922, the formalization of mathematics in terms of axiomatic set theory had emerged. The axioms and rules of inference of this formalism are collectively known as Zermelo-Frankel set theory with the axiom of choice (ZFC). The incompleteness phenomena—assertions which cannot be proved or refuted with ZFC—have not yet necessitated a reassessment of ZFC, but the twenty-first century may see debate on which axioms and rules of inference should be allowed. This talk will provide a historical perspective on the incompleteness phenomena.
Introduced by Saunders Mac Lane, former President of the AMS.



Benedict H. Gross
Professor of Mathematics
Harvard University
Ph.D., Harvard University, 1978

8:30 a.m.
Wednesday, August 10

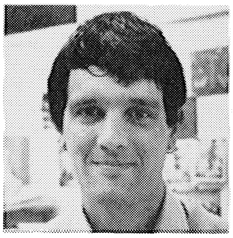
Modular forms and elliptic curves
This talk will survey some major developments in the theory of elliptic curves. The theory of elliptic functions and modular forms, created in the nineteenth century, concerns the real and complex solutions of cubic equations and their moduli. In the last fifty years, the original arithmetic viewpoint has once again emerged. The problem of counting the number of solutions (mod p) to equations with integral coefficients is related to certain Fourier expansions in the classical theory of modular forms. This relationship has led to some progress on the problem of constructing rational points.
Introduced by John T. Tate.



Joseph Harris
Visiting Scholar in Mathematics
Harvard University
Ph.D., Harvard University, 1977

9:45 a.m.
Wednesday, August 10

Developments in algebraic geometry
One of the oldest branches of mathematics, algebraic geometry is concerned with the geometry of curves, surfaces, and higher-dimensional objects defined by polynomial equations—conic sections, quadric surfaces, and so on. Over the last two centuries, algebraic geometry has undergone a series of transformations in which its basic objects of study were redefined, the most recent being the introduction of the concept of "schemes." This talk will describe these stages in the evolution of the subject and indicate how they arose as outgrowths of classical problems.
Introduced by Phillip A. Griffiths.



Roger E. Howe
Professor of Mathematics
Yale University
Ph.D., University of California, Berkeley, 1969

2:00 p.m.
Wednesday, August 10

A century of Lie theory
The subject called Lie theory (the study of Lie groups, Lie algebras, algebraic groups, and their applications) is, like the AMS, just about one hundred years old. In that century, Lie theory has established itself as a central area of mathematics, using tools from many sources and having implications for many other fields. This talk will attempt to give a feeling for the diversity of applications of Lie theory and for the rich internal structure that supports the applications.
Introduced by George Mackey.

Vaughan F. R. Jones
Professor of Mathematics
University of California, Berkeley
Ph.D., Université de Genève, Switzerland, 1979

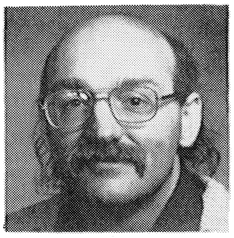
3:15 p.m.
Wednesday, August 10

A von Neumann algebra excursion: From quantum theory to knot theory and back
A surprising result in von Neumann algebras suggested representations of the braid group into an abstract algebra discovered in statistical mechanics. The result allows one to associate to each braid a number which turns out to depend only on the knot obtained by closing the braid. The resulting new knot invariant stimulated the discovery of many more such invariants. These invariants are being used to study the way enzymes "untie" knotted strands of DNA in the process of replication.
Introduced by Joan S. Birman.

Victor G. Kac
Professor of Mathematics
Massachusetts Institute of Technology
Ph.D., Moscow State University, 1968

4:30 p.m.
Wednesday, August 10

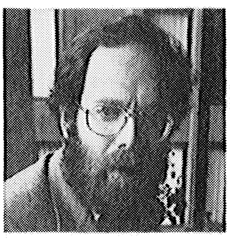
Modular invariance in mathematics and physics
This talk will focus on some beautiful, recently discovered connections between the representation theory of infinite dimensional Lie algebras and the theory of modular functions, and on related progress in theoretical physics. The basic examples covered will be: affine Kac-Moody algebra, the central extension of the loop group of a compact Lie group; and Virasoro algebra, the central extension of the Lie algebra of vector fields on the circle. The "modular invariant" representations of these algebras have been playing a fundamental role in recent developments of conformally invariant quantum field theories and in string theory.
Introduced by Nathan Jacobson, former President of the AMS.



Andrew J. Majda
Professor of Mathematics
Princeton University
Ph.D., Stanford University, 1973

8:30 a.m.
Thursday, August 11

Mathematical fluid dynamics: The interaction of nonlinear analysis and modern applied mathematics
The rapid evolution of applied mathematics through large-scale computation reveals new fluid flow phenomena that are far beyond the capability of experimental measures. To explain and control these complex phenomena, new mathematical ideas from nonlinear analysis, differential equations, probability theory, and geometry must interact with computational methods and more traditional tools of applied mathematics. This talk will present a survey of several examples of this new mode of interdisciplinary research in mathematical fluid mechanics.
Introduced by Peter D. Lax, former President of the AMS.



Charles S. Peskin
Professor of Mathematics
Courant Institute of Mathematical Sciences,
New York University,
Ph.D., Albert Einstein College of Medicine, 1972

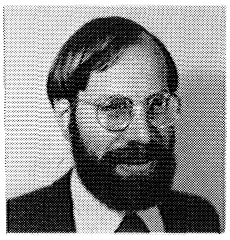
9:45 a.m.
Thursday, August 11

Mathematics and computing in physiology and medicine: Examples from the past, present, and future
The examples considered are the Hodgkin-Huxley equations for the nerve impulse, computed tomography, a mathematical model for blood flow in the heart, and the robotics of large biological molecules. Computation is a key ingredient in all of these examples, and future success is tied to the development of large-scale computers and efficient numerical algorithms.
Introduced by Cathleen S. Morawetz.

Dennis P. Sullivan
Professor of Mathematics
Graduate School and University Center,
City University of New York,
Ph.D., Princeton University, 1966

2:00 p.m.
Thursday, August 11

Progress on the renormalization conjectures in dynamical systems
Computation has led theoretical physicists to the discovery that, in certain dynamical systems, the geometrical structure at successively smaller scales is asymptotically constant. Moreover, the structure is universal in the sense that inequivalent systems have the same limiting structure. This talk will summarize the progress in the theoretical understanding of this numerical discovery.
Introduced by Stephen Smale.



Robert E. Tarjan
James S. McDonnell
Distinguished University Professor
of Computer Science
Princeton University and Distinguished Member
of Technical Staff AT&T Bell Laboratories
Ph.D., Stanford University, 1972

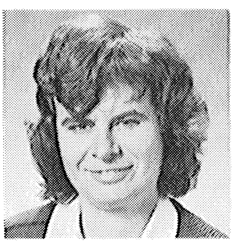
8:30 a.m.
Friday, August 12

Mathematics in computer science
This talk will explore the interdependencies between mathematics and computer science as illustrated in the variety of mathematical ideas used to derive results in computer science theory and the use of computation in the proof of mathematical theorems.
Introduced by Ronald L. Graham.

William P. Thurston
Professor of Mathematics
Princeton University
Ph.D., University of California, Berkeley, 1972

9:45 a.m.
Friday, August 12

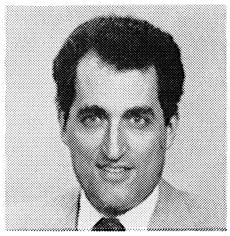
Three-dimensional geometry and topology
Three dimensions is the crossroad for geometry and topology. In dimensions higher than 3, topology becomes much more arbitrary, while geometry becomes much more restricted and rigid. In dimensions lower than 3, topology is more limited, while geometric constructions are more flexible. This talk will describe several instances of the close match between the geometry and topology of 3-dimensional objects, including the theory of polyhedra, the theory of knots, and the theory of 3-dimensional manifolds.
Introduced by Lipman Bers, former President of the AMS.



Karen K. Uhlenbeck
Professor of Mathematics
University of Texas at Austin
Ph.D., Brandeis University, 1968

11:00 a.m.
Friday, August 12

Instantons and their relatives
Instantons are geometric objects which were discovered by theoretical high energy physicists as a result of failed attempts to understand strong interactions. The instanton equation—of which instantons are solutions—derives from the nonlinear version Maxwell's equations formulated by Yang and Mills in 1954. The importance of the instanton equation in mathematics was recognized only in the past decade. Vortices and monopoles are only two of the many related geometric objects having elegant, interesting, and useful mathematical properties. This talk will attempt to describe some of the more colorful properties and uses of instantons and some conjectures for the future.
Introduced by Shiing S. Chern.

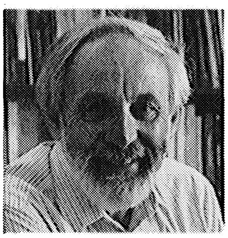


Edward Witten
Professor of Physics
Institute for Advanced Study
Ph.D., Princeton University, 1976

2:00 p.m.
Friday, August 12

Quantum field theory and Donaldson polynomials
When Simon Donaldson initiated a program of using the self-dual Yang-Mills equations to study smooth four-manifolds, the relationship of his work to physical ideas was something of an enigma. Since then, it has become clear that relativistic quantum field theory provides a very natural setting for understanding Donaldson theory and its relationship to Floer theory, elliptic cohomology, conformal field theory, and possibly to other subjects, including string theory and the Jones polynomial. This talk will survey some of these developments.
Introduced by Clifford Taubes.

AMS-MAA Invited Addresses
By invitation of the AMS-MAA Joint Program Committee, the following speakers will speak on the history and development of mathematics.



Raoul H. Bott
William Caspar Graustein Professor of Mathematics
Harvard University
D.Sc., Carnegie Institute of Technology, 1949

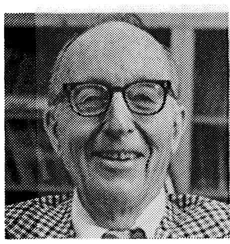
11:00 a.m.
Tuesday, August 9

The topological constraints on analysis
This topic has been at the center of one of the two great American schools of topology. Some of its achievements during this century will be discussed.
Introduced by Andrew M. Gleason.

Peter D. Lax
Professor of Mathematics
Ph.D., New York University, 1949

11:00 a.m.
Wednesday, August 10

Mathematics: Applied and pure
In this century, some have viewed mathematics as separated into pure and applied. Today more and more mathematicians realize that mathematics does not "trickle down" to application areas, but is an equal partner with other sciences. Modern computers have linked mathematics with other sciences.
Introduced by George Daniel Mostow.



Saunders Mac Lane
Professor Emeritus, University of Chicago
Ph.D., University of Göttingen, 1934

11:00 a.m.
Thursday, August 11

Some major research departments of mathematics
In the last century, the development of mathematics has been led by a number of outstanding research departments. The tradition was developed in the U.S. by Moore, Birkhoff, Veblen, Stone, and others. This talk will describe several mathematics research departments.
Introduced by Leonard Gillman, President of the MAA.