The Princeton University Bicentennial Conference on the Problems of Mathematics [1]

Foreword

The forward march of science has been marked by the repeated opening-up of new fields and by increasing specialization. This has been balanced by interludes of common activity among related fields and the development in common of broad general ideas. Just as for science as a whole, so in mathematics. As many historical instances show, the balanced development of mathematics requires both specialization and generalization, each in its proper measure. Some schools of mathematics have prided themselves on digging deep wells, others on excavation over a broad area. Progress comes most rapidly by doing both. The increasing tempo of modern research makes these interludes of common concern and assessment come more and more frequently, yet it has been nearly fifty years since much thought has been broadly given to a unified viewpoint in mathematics. It has seemed to us that our conference offered a unique opportunity to help mathematics to swing again for a time toward unification. For this, could a better way be found than the bringing together of a small and diversified group of capable and active workers and their description to one another of their current research and, above all, the problems facing them, whether close at hand or in the middle distance? Mainly owing to the unremitting efforts of all concerned, the response has been most encouraging, our basic purpose has been amply fulfilled, and decided success achieved. It is our hope that other groups will carry the trend forward, particularly in the fields which we could not cover.

Time was finite, and we were forced to some omissions. Applied mathematics, because of its wide ramifications into many sciences, could not, we thought, be treated as one field; at any event, we would be concerned with

[1] Reprinted from *Problems of Mathematics*, Princeton University Bicentennial Conferences, Series 2, Conference 2.

its unifying spirit, pure mathematics. Yet, as the summary shows, applied mathematics still makes its vitalizing contributions.

Owing to the spiritual and intellectual ravage caused by the war years, it seemed exceedingly desirable to have as many participants from abroad as possible. As the list of members shows, considerable success was attained in this. Our conference became, as it were, the first international gathering of mathematicians in a long and terrible decade. The manifold contacts and friendships renewed on this occasion will, we all hope, in the words of the Bicentennial announcement, "contribute to the advancement of the comity of all nations and to the building of a free and peaceful world."

SUMMARY

The account of the sessions which follows is informal and subjective—it tries to reproduce the atmosphere of each session, the current of the discussion and the main points of interest. We feel that this sort of report will be most helpful to mathematicians at large, particularly since the details will appear in book form under the same title. In some cases the requirements of space have kept us from reporting active and exciting discussion, yet we hope that this brief report will go far in giving to all mathematicians much of the flavor and spirit of the conference.

The completeness and clarity of the reports owes much to the Reporters for the individual sessions, the responsibility for the errors and omissions is mine.

ALGEBRA

The discussion in this, the first session, could not find time to cover all of algebra. Two main lines can be seen, however, one the generalization of known results with an eye toward increasing their scope and learning more of their inner meaning—this going on at widely different levels of abstractness—and the other the continuation along classical lines, represented by Brauer's imposing advance.

The contrast between the present conference emphasizing discussion and Harvard's more formal Tercentenary of ten years ago was noted by Birkhoff. Defining algebra as dealing only with operations involving a finite number of elements, he classified the results of the most general algebra into three classes. The first contains "trivial" results. The second those involving the Axiom of Choice ("where the infinite gets into algebra"), such as that every algebra is a subdirect union of subdirectly irreducible algebras. These Birkhoff felt were becoming trivial. The third was illustrated by the theorem that in any algebra containing a one-element subalgebra, whose congruence relations are permutable, we can prove all known variants of the Jordan-Holder Theorem, Remak's "six-way isomorphism theorem" and (assuming finite chains) the theorem that in any two representations of A as a direct union, the factors are pairwise isomorphic (this enormously generalizes the Wedderburn-Remak-Krull-Schmidt Theorem).

This definition of algebra was challenged by Artin—"What about limits? In a valuation theory we have limits, and while the analysts may consider it a breach of faith ... " Birkhoff replied that "I don't consider this algebra, but this doesn't mean that algebraists can't do it." The value of the additional conclusions which can be added to the subdirect representation theorem in special cases by using additional tools, often topological in character, was emphasized by several members. Mac Lane stressed how topology helped to make the representation theorem for Boolean algebras more useful. Dunford stressed the importance of being able to characterize a representing subalgebra as that of *all* elements with a certain property—in analytical applications this often allows one to prove the existence of square roots, which led Stone to say that "It has seemed to me that the words 'all' and 'compact' are vaguely equivalent here." The advantages of having known, simple structures for the subdirectly irreducible algebras were pointed out; thus for Boolean algebras the fact that they are just two-element algebras is of decisive significance.

Rado suggested that the definition of congruence relation might be generalized to include cyclic transitivity and the conjugacy relation in a Peano space. Albert felt that it would be better not to generalize so far, and pointed out the possibilities of shrinkable algebras, a generalization of Jordan algebras where the powers of any element still commute with one another.

The generalized Jordan-Holder theorem was discussed in the light of the hypotheses and proof of Birkhoff on the one hand and Tarski and Jonsson on the other. The need of unifying results was clear to all who took part.

Brauer reported his proof of Artin's conjecture about induced characters, which asserts that characters known to be rational combinations of certain special characters are in fact integral rational combinations. The idea of the proof, and much technique was supplied by the theory of modular characters, but results from this field were not used. From this result it follows, for example: (1) that the zeta-function of a field K normal over k is divisible by the zeta-function of k (in the sense that the quotient of the functions is an integral function); (2) that every representation of a group G can be written in the field of the q-th roots of unity, where q is the least common multiple of the orders of the elements of G; (3) a conjecture of Siegel (1934) concerning the asymptotic behaviour of the class number. Brauer's result represents a decisive step in the generalization of class-field theory to the non-Abelian case, which is commonly regarded as one of the most difficult and important problems in modern algebra.

Artin stated that "My own belief is that we know it already, though no one will believe me—that whatever can be said about non-Abelian class field theory follows from what we know now, since it depends on the behaviour of the broad field over the intermediate fields—and there are sufficiently many Abelian cases." The critical thing is learning how to pass from a prime in an intermediate field to a prime in the large field. "Our difficulty is not in the proofs, but in learning what to prove."

A general Galois theory was characterized by Jacobson as beginning with "some sort of system in the field" and then setting up "some sort of correspondence between the intermediate fields and the subsystems of the system." He presented two such theories, one based on the notion of self-representation, which was introduced by Kaloujnine, and a second, applying to the non-commutative case, which combined self-representations, the allied notion of a relation space and the methods of the structure theory of rings. A problem of Artin concerning the equality of the right and left dimensionality of a division ring is closely connected to these theories, but remains unsettled in the general case. Further open questions are: "Can Hilbert's theory for normal extensions of an algebraic number field (which links up the arithmetical structure with the Galois group) be generalized to the case of an arbitrary finite extension by means of the self-representations of the extension? Can this be extended to include infinite extensions?"

Algebraic Geometry

Here the discussion followed two lines — new, deeper problems for the classical algebraic geometry over the field of complex numbers contrasted with new methods for developing algebraic geometry over abstract fields. Lefschetz stated four problems, three along the first line and the fourth along the other: (1) Study the dimension of linear systems on algebraic varieties; extend the Riemann-Roch theorem to higher dimensions. (2) Can algebraic surfaces be parametrized by automorphic functions (in a way analogous to that of Poincaré for algebraic curves)? The Betti numbers of the fundamental domains of the automorphic functions were suggested as a fundamental tool for disproving this. (3) Is the cubic variety in four-space rational? The rumor that Fano has settled this in the negative was discussed. (4) Examine the standard configurations (e.g. the 27 lines on a cubic surface) over fields of characteristic p. "This one is baby talk. But there are some very deep problems in the same direction. The ambitious young fellows with lots of shoulder muscles could do much worse than to study Max Noether's and Halphen's work on the classification of algebraic curves in a 3-space and try to carry it over to characteristic p."

The next topic was the rank problem for cycles on the orientable manifold defined by an (irreducible) algebraic variety over the complex numbers. The rank of a cycle is determined by the dimension of the smallest algebraic sub-variety which bears a homologous cycle. Necessary conditions were obtained several years ago by Hodge in terms of the vanishing of the integrals of certain closed differential forms over the given cycle. The problem is to show that these conditions are also sufficient.

Hodge pointed out three component parts from which a proof of sufficiency would follow. The central part was the following: If Γ_m is an m-cycle of V_m and every integral (of type zero) of the first kind has zero period on Γ_m, then Γ_m is homologous to a cycle on an algebraic sub-manifold $D_{m_{-1}}$ of V_m. This is unsettled, except in the case $m = 2$ (Lefschetz) and is a problem of classical algebraic geometry. Hodge feels that any solution of this part will, through its technique, provide solutions of the other parts. This initiated an active discussion of the possibilities of giving meaning in the abstract case (including characteristic p) of the topological ideas, such as integral and periods, which play such a central part in the present theory for the classical case. A. Weil's definitions of residues and sums of residues were agreed to represent the best progress to date but the field seemed comparatively untouched.

With the aid of the entire theory of numerical invariants, Castelnuovo and Enriques showed that every algebraic surface (except ruled surfaces) was birationally equivalent to a minimal model—one in which no birational transformation can further shrink a curve into a point. If this were provable without the aid of the theory of invariants, that theory could be greatly simplified. Zariski outlined such a proof, which also applies to the case of characteristic p. "By slipping in exceptional curves in all possible ways we obtain what is probably the worst offender in this respect—the Riemann surface of the field where there is a point for each of the places, each of the valuations of the field. Since I have used this model in my work, I felt that I should work toward a model with the opposite property." The revision of theory allowed by this result seemed to Zariski to offer a possibility of founding a theory for higher dimensional varieties.

The discussion here led Lefschetz to say that "To me algebraic geometry is algebra with a kick. All too often algebra seems to lack direction to specific problems." Birkhoff countered with "If the algebraic geometers are so ambitious, why don't they do something about the real field?" Lefschetz stressed the resemblance to number theory before the utilization of analytical methods. "There are only scattered results and no indication of general methods." An instance of our ignorance is Hilbert's sixteenth problem about the nested ovals, still unsolved.

Differential Geometry

Here the discussion was concerned with isolated topics from differential geometry in the small and a more integrated discussion of differential geometry in the large.

Hlavaty discussed some recent developments in the differential geometry of the lines of a flat projective three-space. These are expounded in full in his recent book ("Differentielle Liniengeometrie," Noordhoff, Groningen, 1945).

Thomas discussed the problem of determining all quadratic first integrals of the differential equations of a dynamic system. The purely geometric analog of this problem was solved by Thomas and Veblen in Thomas's doctoral thesis. They found a finite process to determine the exact number of independent quadratic first integrals. This can be applied to determining when two dynamical systems have the same trajectories (without regard to time parametrization). Thomas suggested that the simultaneous use of several quadratic forms would be useful in attacking other problems in differential geometry. Rado asked for a definition, saying "You spoke of trajectories independent of parameters. I have often tried to find out from physicists or differential geometers what notion of curve they use. Can you tell me here?" This was not followed up, perhaps because there are so many detailed definitions in the literature. Synge asked Thomas whether, given a concrete dynamical system, he could tell whether there is another first integral. "Yes, in theory. In practice I think it could be done without too much trouble—at least to count the solutions, if not to find them."

The discussion then passed to problems "in the large." Here the objective is to relate the local differential properties of a space to the topological properties of the space as a whole. Results in this theory are still fragmentary and few general methods have been developed. Bochner presented certain new relations between the Ricci curvature of a compact Riemann space and the characteristics of vector fields defined over the space. For example he showed that if the Ricci tensor is positive definite, there exists no vector field for which the divergence and curl both vanish. From this it follows that the first Betti number of the space is zero. Many of the results have more extensive form on a complex analytic manifold on which is defined a positive Hermitian metric. Whitehead commented that "This is the first time that anyone has got a topological result out of the Ricci tensor, without using the full curvature tensor." Veblen said that "This is the direction in which we want to see differential geometry go."

Hopf called attention to little-known but very suggestive results of Preissmann (1943) and Cohn-Vossen (1935–1936). From Preissmann's results we learn that a three-dimensional torus cannot have curvature everywhere < 0 (from Bochner's result it cannot be everywhere > 0). Also the product of two closed surfaces of genus at least two cannot have curvature everywhere < 0 although the curvature can be everywhere ≤ 0. Cohn-Vossen has made an essential extension of the Gauss-Bonnet equality for closed two dimensional surfaces to an inequality for open two-dimensional surfaces. These inequalities relate the total curvature of the surface (a local property) to the Euler-Poincaré characteristic (a topological property of the surface as a whole). The extension of the Gauss-Bonnet equality to closed n-dimensional Riemann surfaces has been accomplished by Allendoerfer and Weil; but similar extension of Cohn-Vossen's inequality remains to be carried out.

MATHEMATICAL LOGIC

Here the discussion revolved about a single broad topic—decision problems. The notion of a decision method is a formalization of the classical notion of an algorithm. The known equivalence of Turing's "computability," defined in terms of a very general computing machine. Herbrand and Gödel's "general recursiveness," and Church's "λ-definability" has led to general agreement on the natural formalization of the notion of an algorithm. More and more decision problems are being shown to be unsolvable, in the sense that there exists no algorithm with the required properties. While the general mathematician has to regard this as somewhat negative progress, it is real progress, not only for mathematical logic but for other fields of mathematics as well.

The nearest approach to a proof of unsolvability for a problem of importance in other fields is the recent theorem of Post that the word problem in semi-groups is insoluble. Such results have encouraged mathematical logicians to go further and try to show that problems of a more standard mathematical character are insoluble. In his summary, Church suggested the word problem for groups and the problems of giving a complete set of topological invariants for knots and for closed simplicial manifolds of dimension n as likely possibilities. Theorems are needed to characterize or provide criteria for the distinction between solvable and unsolvable decision problems (though to find a decision method is no doubt itself an insoluble decision problem).

Closely related to the decision problem for theories is the question of whether or not particular questions are decidable in a given theory. It was pointed out that in particular the Riemann Hypothesis might be undecidable in a particular theory, but that the flexible position of the general mathematician would prevent its ever becoming demonstrably undecidable for him.

The analogy between generally recursive sets of integers and Borel sets of real numbers was stressed by Tarski, who pointed out the possibilities of proving analogous theorems and of developing a single theory to include both as special cases. Tarski then surveyed the status of the decision problem in various logical fields: sentential (propositional) calculus, predicate (functional) calculus, many-valued systems of sentential calculus, number theory, analysis, general set theory, and various abstract algebraic systems. In all of these, even in two-valued sentential calculus, where we would like to be able to decide when a set of formulas is an adequate axiom system, he pointed out open, important problems.

No formal system, with the usual restrictions, which is strong enough to deal with the arithmetic of integers can be complete (Gödel 1931), it must contain undecidable propositions. This led Gödel to propose a particular tremendous enlargement of the notion of a formal system—which would allow uncountably many primitive notions and allow the notion of an axiom to depend on the notion of truth. "I do not feel sure that the set of all things of which we can think is denumerable."

This led to a spirited discussion led by Church and Gödel, which centered on the non-mathematical questions of what could reasonably be called a "proof" and when a listener could "reasonably" doubt a proof. From a psychological point of view the discussion resembled the classical debates on intuitionism to a remarkable degree; Church arguing for finiteness and security and Gödel arguing for the ability to obtain results.

Kleene discussed the limitations which general recursiveness may place on quantitative proof. McKinsey discussed the criticisms of general recursiveness as the formalization of the notion of an algorithm.

Quine proposed to evade the undecidability of arithmetic by studying a restricted arithmetic without quantifiers, which might have a decision method. It was pointed out that, in particular, Fermat's Last Theorem could be expressed in such a system. The corresponding problem for a partial system of real numbers without quantifiers was proposed as an open problem.

The subject of non-classical logic recurred throughout the session, with relatively favorable words for the quantum-mechanical system of Birkhoff and von Neumann. Tarski felt that "The system of von Neumann and Birkhoff seems to me to be the most interesting of these (non-classical logics), and the only one which has any chance to replace our customary two-valued logic, since it is the only one which has arisen from the needs of science." Rosser discussed the problems involved in applying a many-valued logic, for example Reichenbach's, to all the steps which lead up to quantum mechanics, including truth-function theory (propositional calculus), quantification theory (functional calculus), set theory, theory of the positive integers, theory of real numbers, theory of limits and functions, theory of Hilbert space, theory of quantum mechanics. The initial steps have shown a great tendency of the theory to ramify single notions in the classical theory corresponding to more and more distinct notions as one passes to more and more complex theories. Gödel proposed using the two kinds of logic, each in its place. Church would accept this proposal for consideration, only if a single logistic system were constructed providing syntactical criteria by which the place of each kind of logic is fixed. He felt that all non-classical logics faced great difficulties in such applications.

Topology

The discussion centered around two main topics: (a) groups of transformations, (b) classification of homotopy classes of maps, fibre bundles and related questions. The detailed discussion was extremely active and so wide-ranging that it cannot be covered here in every detail.

The discussion of (a) was initiated by an account by Montgomery. "Let G be a topological group and M a topological space." "A function $f(g;x) = g(x)$ defined and continuous on the product of G and M with values in M gives rise to a transformation group of M (or a representation of G) if (a), for a fixed $g, g(x)$ is a homeomorphism of M onto itself and (b), $g_1[g_2(x)] = (g_1 g_2)(x)$ for all g_1, g_2, and x." "The transformation group is called effective if only the identity leaves all of M fixed." "The group G is usually taken to be compact or locally compact, and M is taken to be a manifold." There are problems on various levels of generality mainly singled out by whether continuity alone, or a certain amount of differentiability,or analyticity is assumed for $f(g;x)$. If only continuity is assumed, some of the main questions are closely related to the fifth problem of Hilbert, a central conjecture being that any compact group which acts effectively on a manifold must be a Lie group. This is true for a compact connected group acting on a three-dimensional manifold and is true for any dimension with the additional assumption that each individual transformation is of class C^1. The conjecture also holds for a locally compact group if each transformation is of class C^2. In the discussion several problems which lie on the path towards a solution of the main conjecture were stated. Can a p-adic group operate effectively on a manifold? Does every periodic transformation of a Euclidean space admit a fixed point?

The discussion of the second main topic was initiated by Steenrod. The general problem consists in finding algebraic methods for the enumeration of the homotopy classes of continuous maps $K^n \to L^m$ where K^n and L^m are polyhedra of dimension n and m respectively. If $L^m = S^n$, the n-sphere, the problem has been solved by Hopf. If $L^m = S^{n-1}$, the $(n-1)$-sphere, the problem has been solved by Pontrjagin and Steenrod using various multiplications in the cohomology theory. The case where $K^n = S^n$ leads to the definition of the n-th homotopy group of L^m (Hurewicz). These two main trends overlap when both K^n and L^m are spheres. We then have the problem of computing the homotopy groups of spheres. Although many efforts were and are directed towards this problem, only very fragmentary information is available. "Here is a complete unknown morass of groups."

The classification of fibre bundles and finding their invariants is closely related in nature and methods. Recent progress and some problems for the future were discussed by Hopf, who also asked whether every abstract cohomology ring is realized by a complex.

On the methodological side two facts were stressed. First, the constant use of obstruction cochains and cocycles with coefficients in relevant homotopy groups. Second, the need of studying not only the groups involved but also the homomorphisms connecting them.

In a final summary J. H. C. Whitehead noted with satisfaction that one of the main trends in present day topology is towards the study of the deeper properties of relatively simple spaces, such as spheres, spaces of line elements on spheres, group spaces, etc. He classified the problems of topology under three broad headings. At one end there are the fundamental definitions and problems, particularly the Poincaré conjecture that the 3-sphere is the only closed 3-manifold whose fundamental group (first homotopy group) is the identity, the basic conjecture that topological equivalence of complexes implies identity of subdivisions and the "word problem." Our knowledge of these matters is practically nil. At the other end there is the theory of homology. Here some of the fundamental problems such as calculating the homology groups, intersection cycles, etc., are completely solved by finite algorithms. In between homology and the ultimate problems of topology comes the homotopy belt. Here there are in general no finite algorithms. However, it is often possible, as in the theory of knots, to find an algebraic expression of a general class of problems, which is so closely related to the geometry and is so amenable to manipulation that it gives a good chance of solving any concrete problem, not too complicated to be stated. Much of the work discussed at the session consists of pressing up from the homology to the homotopy region, partly by the application of the former to the latter. Whitehead also referred to combinatorial theory initiated by Newman, to Reidemeister's *Überlagerung* and to a combinatorial theory with its roots in these two which he is developing. He described this as approaching the problems of homotopy from the other direction.

New Fields

Although this session was entitled "New Fields" the discussion was principally of classical problems related to application, and of the need and feasibility of revitalizing work in these fields. An exception must be made for Wiener's discussion of communication problems. This is not to be taken as a sign that there are no new fields in mathematics, but rather as an indication that new fields grow as part of an old field, or of a particular application, until they are so large as to be no longer new.

Active discussion first took place about the importance of rigorous theorems in applicable potential theory. The fact that all these theorems held in their integral form satisfied Weyl and Courant, since one really needs only the force on a small body and not on a point. Birkhoff felt that "It is a worthwhile mathematical problem to show that what is obvious is mathematically deducible," while Weyl commented that "It depends on the kind of mathematical deduction."

Evans discussed the status of multiple valued harmonic functions in three-dimensional space, which are of distinct importance in the theory of diffraction. Murnaghan outlined a non-linear theory of elasticity with finite displacements. The relation of this theory to the theory of plasticity was discussed, and it was pointed out that it could not explain the observed phenomena of hysteresis. Synge pointed out how the consideration of vectors, whose components lie in a function space provides both geometric intuition and a workable method of computing the size of the error committed by certain approximate solutions of boundary value problems. The relation of this to the Rayleigh-Ritz procedure was discussed.

Wiener called attention to the problems of communication of information, and to their general nature, which has "nothing to do with electricity, passive networks, or electrical engineering." "The theories of control engineering and communication engineering are one and the same—(and) are indistinguishable from the theory of time series in statistics." He emphasized the problems arising in this field in modern computing machines and the human nervous system (mathematical neurophysiology), where many problems need solution. The relation between the notions of amount of information in this theory and in R. A. Fisher's theory of statistical inference was briefly discussed.

von Neumann pointed out that the success of mathematics with the linear differential equations of electrodynamics and quantum mechanics had concealed its failure with the nonlinear differential equations of hydrodynamics, elasticity, and general relativity. He emphasized the likelihood that we face singularities of new types, not foreshadowed by those of functions of a complex variable. In the general relativity theory of a mass particle, there is an apparent singularity near the particle whose position depends on the coordinate system. We do not know what the equivalent of the "objective" theory of function-element continuation is in this case. In non-viscous gas dynamics the "natural" boundary conditions generally do not allow solutions unless certain discontinuities are allowed, these on the other hand force one to violate certain integrals (entropy), and introduce difficulties with uniqueness.

Further, we have no adequate theory of the interactions of two shock waves! In turbulence, a symmetric problem is dominated by unsymmetric solutions!

Mathematical Probability

Cramér surveyed many of the problems related to sums of random variables. A first group of problems concerned finite sums of independent random variables and the pathological cases already known. He emphasized the lack of general knowledge of the situation. For example, which infinitely divisible distributions can be represented with an indecomposable factor?

The Central Limit Theorem for independent variables seems to be in satisfactory shape, but we lack much information about when it is valid for sums of dependent variables. This is the simplest case of the general problem of limiting distributions, about which we know so little.

Stochastic processes were discussed more actively than any other topic; Cramér went over the general state of the theory, pointing out many directions in which our knowledge is incomplete. Doob called attention to the advantages of actually considering the specific functions of the stochastic process, saying "In the good old days, a lot of people worked in probability, but they got out of this field and into differential equations, or integral equations as soon as possible. This is not a criticism, but only a description." Feller pointed out that the results already known for Markoff processes are valuable tools in studying the functional equations of many types, which are known to hold for such functions associated with the process as the probability of a given transition in a fixed time interval. These equations may be, in particular, (i) a system of infinitely many differential equations, (ii) a difference-differential equation, (iii) infinite systems of difference-differential equations. "It is theoretically possible to consider any stochastic process as a Markoff process, but in a perfectly crazy space. The question is, does it help us or hurt us?" The problems of multiple renewal theory seem to require much further work along the lines of classical integral equation theory.

Hotelling noted the absence of statistics, as contrasted with probability theory, from the discussion, and emphasized the need for a complete and careful study of the estimation problem. The fact that a finite number of samples of finite length must be used to estimate an infinite process is sure to lead to new difficulties and interesting results. Wiener's experience with actual estimation in stationary processes was closely parallel to the experience of statisticians in estimating distributions of a single variable—it was hard, but feasible, to estimate general parameters, but it was much more difficult to estimate detailed qualitative characters of processes, such as would be given by higher moments, for example.

Kac discussed some problems connected with the central limit theorem, and pointed out that, since for a stationary Gaussian process one can define the likelihood of a path, we could ask "Is there a path of maximum likelihood?" He also proposed the problem of the Brownian motion of a stretched string.

The difficulties of further advances in some fields of stochastic processes were sharply outlined by Kac's statement, partly in jest, that "It seems to me that Doob's discussion could be summarized by saying that people were using difference equations to calculate probabilities when they didn't know what probability was, and that now we know what probability is, but can't calculate it." The discussion repeatedly fringed on a discussion of the *ad hoc* nature of Doob's powerful methods, but this seemed to lie just outside the purely mathematical discussion.

Returning to more finite problems, Neyman raised a representation problem for sequences, and Wald discussed the problem of the maximum absolute value of a partial sum of a random series. This problem, of considerable applicational interest, is just being effectively attacked. The use of higher moments and analytical techniques was proposed by Wiener.

Analysis

The first group of problems discussed arose from the study of partial differential equations. Classical potential theory is concerned with solutions of Laplace's equation, a special equation of elliptic type. Riesz described the modern development of potential theory, first for Euclidean space, and then for the so-called Lorentz space in which the square of the "distance" between points $(x_1, \dots, x_m)$ and $(y_1, \dots, y_m)$ is defined as $(x_1 + y_1)^2 - (x_2 - y_2)^2 \cdots (x_m - y_m)^2$. In the first, or elliptic, case, the classical Newtonian potential depending on the inverse $(m - 2)$th power of the distance, where the dimension m is at least 3, is replaced by one depending on the inverse $(n - a)$th power, where a is between 0 and 2. Boundary value problems in this generalized potential theory introduce new problems which did not appear in the classical theory; the methods developed for their solution throw new light on classical potential theory. Riesz stressed the idea, which is of particular importance in the new theory, that it is more satisfactory to define the Green's function as a potential than as the solution of a boundary value problem. In Lorentz space, the potentials appear as generalizations of the Riemann-Liouville integrals of fractional order. Riesz showed how the potentials lead in a very direct way to the solution of Cauchy's boundary value problem for the m-dimensional wave equation. In discussion, Hille emphasized similarities between the Riesz Lorentz potentials and the Riemann-Liouville integrals. He described some of their properties, regarding them as operators belonging to semigroups of operators over Banach spaces.

The second group of problems were inspired by the general problem of representing functions by series. Zygmund described the two major outstanding problems of the theory of trigonometric series. For Fourier series in one variable the problem of representing a given function may be considered solved in the sense of summability (the present popularity of summability problems seems unfounded); however, even this is far from true for multiple series. For convergence, on the other hand, even the problem of whether the Fourier series of a continuous function must have points of convergence is unsolved and almost no progress has been made since Riesz's report in 1913. A solution would give new insight into the structure of functions of a real variable. Several conjectures were mentioned, but "the theory of trigonometric series is one of the worst places to have conjectures. History teaches that almost all conjectures were proved false." The second problem is that of characterizing sets of uniqueness for Fourier series; sets such that two trigonometric series which converge to the same sum outside the set, they are necessarily identical. Rajchman's conjecture about sets $H\sigma$ is still unsettled. The properties of sets of uniqueness are known to depend in a sense on their arithmetic structure. In this connection the properties of a special class of algebraic integers are relevant. In discussion, Salem stated several further problems involving these algebraic integers. Walsh called attention to a problems analogous to Fourier series problems but related to a general closed curve as Fourier series are related to a circle. Wiener stressed the importance of these problems for open curves as in the Nyquist diagram. Birkhoff brought out the problem of sets of uniqueness for solutions of partial differential equations, in particular for axially symmetric harmonic functions.

The third group of problems centered about the distribution of the values of analytic functions of a complex variable. Ahlfors mentioned that since the advances resulting from the modern introduction of topological and differential geometric methods into the problems related to Picard's theorem, little progress has been made, either with results relating specifically to the problem of the distribution of values or with the so-called type problem, the problem of classifying Riemann surfaces with respect to their ability to be mapped onto either a circle or the finite plane. He stressed the importance for this field of the theory of the conformal mapping of multiply connected domains. Further progress in the theory of entire functions seems to depend on progress in such conformal mapping problems; the first step appears to involve the solution of some new extremal problems. Robinson discussed the fate of the maximum modulus theorem in a multiply connected domain. Boas gave an account of problems involving entire functions of exponential type. Here the slow rate of growth of the function allows one to make quite precise statements about the restrictions imposed on the function by its special behavior in various respects. Levinson called attention to a new technique introduced by A. Selberg in the study of the distribution of the zeros of the Riemann zeta function, which may have consequences in analytic number theory.

Analysis in the Large

Here the discussion centered on the choice of method—many of the problems of analysis in the large can be attacked by at least two different techniques, and it is of central importance to choose the approach which will give the best results with the least difficulty. In opening the session, Morse emphasized that the division of the session into differential geometry in the large, applied mathematics in the large, and equilibrium analysis in the large was "all for convenience and has nothing to do with essential differences."

In outlining the problems facing differential geometry in the large, Hopf reemphasized the importance of the work of Cohn-Vossen (1935–1936) on open surfaces and the need of extending it to higher dimensions. "Can the result that on a complete surface of the type of the plane with $K > O$ every geodesic goes to infinity be generalized? Can the existence and uniqueness theorems for a complete surface with $K > O$ of the type of the sphere as subspaces of E^3 be generalized by omitting $K > O$?" "In which E^m can the hyperbolic plane be imbedded?" "Given a closed differentiable manifold of dimension n, for what k can we find k functions $x_1, x_2, \dots, x_k$ whose gradients are of dimension n?" "On $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, there are given k continuous functions without common zero. Can they be extended to the interior of the sphere without common zero?" Recent progress has been made on this last topological problem. The necessary and sufficient condition can be given in terms of differential geometry (by the vanishing of an integral) when $n = k$, and, as J. H. C. Whitehead reported, when $n = 4, k = 3$.

Although he admitted that some may think it a dream, Allendoerfer said that a program of connecting differential invariants with topological invariants had been in the minds of many. For this program it seems that the invariants should be integers, although "we know that topologists like groups." Hedlund discussed the problems of geodesic flows on a manifold, which arose from the consideration of dynamical systems. "Is a metric on the torus either flat or such that each geodesic has two conjugate points?" "If there are no conjugate points, is there metric transitivity?" "If there is negative curvature (not necessarily constant) is there mixture?" Hodge considered a complex manifold of two complex dimensions with coordinates z_1, z_1 (all indices run through 1, 2) on which functions a_{ij} satisfy

$$\frac{\partial a_{ij}}{\partial z_k} = \frac{\partial a_{kj}}{\partial z_i}, \frac{\partial a_{ij}}{\partial z_k} = \frac{\partial z_{ik}}{\partial z_j},$$

which are the conditions of integrability of

$$\frac{\partial^2 \Phi}{\partial z_i \partial z_j} = a_{ij}.$$

"When does there exist a solution Φ, all of whose singularities lie on an algebraic surface?" He pointed out that few would realize the equivalence of this with the problem that he had stated in the algebraic geometry session.

The need of applied mathematics for existence theorems and uniqueness theorems as well as for computing procedures was stressed by Courant. "If you cannot prove the existence of the solution, then you are not certain that your mathematical model is at all a satisfactory representation of physical reality." "If you cannot prove uniqueness, but can prove the existence of solutions, then your results are unsatisfactory, since, at best, each individual solution may or may not be followed by physical reality." The history of gas dynamics and the phenomenon of shock waves gives much support to these results. The presence of shock waves indicates the utter failure of differential equations alone to describe an idealized gas without friction or heat conduction, for, contrary to Lord Rayleigh's belief, there are no shock-free solutions for even the elementary problems. "We are all used to problems where, when we let a parameter tend to zero, the solution will tend to the solution for the zero value of the parameter. But this is exactly what does not happen." Shiffman pointed out that the differential equations for imbedding the hyperbolic plane into three-space are nonlinear and the analogy is strengthened by Hilbert's proof of nonimbeddability, which shows that singularities must arise.

The calculus of variations and the Schauder-Leray method are the two broad tools in much of analysis in the large. There was much discussion of their properties and contrasts. Shiffman asked for a method of applying Schauder-Leray theory directly to a class of analytic functions. He also hoped that the two methods could be brought closer together "so that they may alter and improve each other, and also so that each may fill out the gaps in the scope of the other." Uniqueness is harder to establish than existence by Schauder-Leray methods—Weyl felt that this was distinctive. Morse felt that the distinction between fixed point and extremal problems was general, saying: "The Schauder-Leray theory treats more general problems, with less powerful tools, thus obtaining less complete results." Thus those problems which admit a variational form, e.g. those with symmetric kernels, are best treated by extremal methods.

Radó pointed out that the right restriction for all single integral problems was absolute continuity and that is exactly what was suggested by the length integral. This was as if "by looking at the first two terms of an infinite series, one determined its convergence." He suggested that essential absolute continuity might play the same role in two, and perhaps even in higher dimensions.

In opening the topic of equilibrium analysis in the large, Morse stressed that the study of the behavior of two or more functions can always be reduced to the study of one function with general boundary conditions. "I feel that this is a better way to do this." This way passes through seven problems, each broad enough to be a program, and all of these need solution for progress on a broad front. 1. Two-point connectivities and generalizations. So far found only for specific manifolds by analytic solution of analytic problems. 2. Incidence of homology classes. Only special cases known. 3. Lower-semicontinuity and upper reducibility. When you use compactness instead of upper reducibility "You are no longer abstracting, you are subtracting—you are throwing away the best part of the calculus of variations." 4. Finiteness of the number of minimal surfaces bounded by the given curve. "I conjecture that an analytic and regular curve in E^3 is spanned by only a finite number of minimal surfaces." (The higher dimensional analog is known to be false.) 5. The derived representation of a given boundary curve. 6. The minimal surface problem on a general R-manifold. 7. The topological foundation of planetary orbits. (This refers to the three body problem with one infinitesimal body.) "The cycles must be relative—they must be loops hanging over ridges and extending down to Dante's Inferno."

McShane discussed the application of L. C. Young's generalized curves, which are "as constructive as the Lebesgue integral, useful, but certainly not perspicuous." He pointed out that, at present, they do not solve the problems of the second variation, or of minimax curves of higher type.

Final Session

At the closing session, a dinner at Procter Hall, the dining hall of the Graduate College, the conference and its guests considered for the time the state of mathematics as a whole and its prospects in the century ahead. As the first speaker, Mac Lane discussed his impressions of the conference and of the state of mathematics in the following vein:

"In almost all the conferences we ran across the phenomenon of someone else moving in. I think this is a significant aspect of this conference. When you set out to solve problems in mathematics, even in nicely labelled fields, they may well lead you into some other field. To cite some instances: The logicians moved in on the algebraists, the topologists moved in on the differential geometers (and vice versa), and the analysts moved in on the statisticians.

"The problem is that of communication between different mathematicians in different fields. The day is no longer here when all mathematics can be done under the old ideal of the universal mathematician who knows all fields.

Problems must rather be done by mathematicians in different fields who understand what is being done in neighboring fields and who are able to communicate with each other in the presentation of their results, in the understanding of what that presentation means, and in the appreciation of the significance and purpose behind the problems which they are endeavoring to solve. It is my opinion that this conference at Princeton has contributed much to the solution of this problem of the communication of mathematics."

Weyl reviewed the development of mathematics in his lifetime, and its relation to his own work. He expressed his tendency toward intuitionism, saying:

"In his oration in honor of Dirichlet, Minkowski spoke of the true Dirichlet principle, to face problems with a minimum of blind calculation, a maximum of seeing thought. I find the present state of mathematics, that has arisen by going full steam ahead under this slogan, so alarming that I propose another principle: *Whenever you can settle a question by explicit construction, be not satisfied with purely existential arguments.*"

And again, more generally: "At a conference in Bern in 1931 I said: 'Before one can generalize, formalize, or axiomatize, there must be a mathematical substance. I am afraid that the mathematical substance in the formulization of which we have exercised our powers in the last two decades shows signs of exhaustion. Thus I foresee that the coming generation will have a hard lot in mathematics.' The challenge, I am afraid, has only partially been met in the intervening fifteen years. There were plenty of encouraging signs in this conference. But the deeper one drives the spade the harder the digging gets; maybe it has become too hard for us unless we are given some outside help, be it even by such devilish devices as high-speed computing machines."

In conclusion, Stone expressed his firm faith that mathematics would continue to grow as well and as fast as of old, now that the ten years under the shadow of war were past.

Judged from the interest of the conferees, these were the trends and results of most interest for the present and near future:

1. The discussion about general algebra—were limits and topological methods really a part of algebra? Many vital results seemed to need these other methods—if they were to be *defined* out of algebra, then algebra would lose much power.

2. Brauer's result on induced characters, which some feel to be the most important in algebra for the last ten years, and which may open the way to non-Abelian class-field theory.

3. The renaissance of interest in algebraic geometry over the field of complex numbers, side by side with a continuing interest in the abstract case.

4. The tendency toward a welding together of differential and algebraic geometric, topological, and Hermitian methods and problems—the trend toward a unified study of manifolds, vector fields, imbedding and critical points.

5. The liveliness of mathematical logic and its insistent pressing on toward the problems of the general mathematician.

6. The return of topology to the study of geometric and combinatorial problems of simple objects, in whose light the recent period of abstraction seems to have been a search for new tools and the natural, exuberant generalizations of a field engaged in finding itself.

7. The tendency for the applications to stimulate what may some day be new fields and to force the reawakening of old, incomplete fields.

8. The role of stochastic processes in the future of mathematical probability and statistics. This seems to be the main subject for advances in pure theory and a major subject for advances in the applications.

9. The absence in analysis of a central, guiding theme, or even of a few such themes. This field seems to be going through a period of wide ramification, perhaps in preparation for new syntheses.

10. The remarkable size and unity of analysis in the large, which yet may well absorb much of the activities and interests now thought of as belonging to topology, algebraic geometry and differential geometry, as well as those classically considered as belonging to analysis.

Guests of the University Who Participated

L. V. Ahlfors, Harvard University
A. A. Albert, University of Chicago
J. W. Alexander, Institute for Advanced Study
C. B. Allendoerfer, Haverford College
G. Ancochea, University of Salamanca, Spain
E. G. Begle, Yale University
G. Birkhoff, Harvard University
R. P. Boas, Mathematical Reviews, Brown University
H. F. Bohnenblust, California Institute of Technology
K. Borsuk, University of Warsaw, Poland
R. Brauer, University of Toronto, Canada
S. S. Cairns, Syracuse University
L. F. Chiang, Academia Sinica, Shanghai, China
I. S. Cohen, University of Pennsylvania
R. Courant, New York University
H. Cramér, University of Stockholm, Sweden

P. A. M. Dirac, University of Cambridge, England
J. L. Doob, University of Illinois
N. Dunford, Yale University
S. Eilenberg, Indiana University
A. Einstein, Institute for Advanced Study
L. P. Eisenhart, Princeton University
G. C. Evans, University of California
W. Feller, Cornell University
K. Gödel, Institute for Advanced Study
O. G. Harrold, Princeton University
G. A. Hedlund, University of Virginia
T. H. Hildebrandt, University of Michigan
E. Hille, Yale University
V. Hlavaty, Charles University of Prague, Czechoslovakia
G. P. Hochschild, Harvard University
W. V. D. Hodge, University of Cambridge, England
H. Hotelling, University of North Carolina
H. Hopf, Federal Institute for Technology, Zurich, Switzerland
L. K. Hua, National Tsing Hua University, Peiping, China
W. Hurewicz, Massachusetts Institute of Technology
N. Jacobson, Johns Hopkins University
M. Kac, Cornell University
S. C. Kleene, University of Wisconsin
J. R. Kline, University of Pennsylvania
N. Levinson, Massachusetts Institute of Technology
S. Mac Lane, Harvard University
W. Mayer, Institute for Advanced Study
J. C. C. McKinsey, Oklahoma Agricultural and Mechanical College
E. J. McShane, University of Virginia
D. Montgomery, Yale University
M. Morse, Institute for Advanced Study
F. D. Murnaghan, Johns Hopkins University
J. von Neumann, Institute for Advanced Study
M. H. A. Newman, Victoria University of Manchester, England
J. Neyman, University of California
W. V. Quine, Harvard University
H. Rademacher, University of Pennsylvania
T. Radó, Ohio State University
M. Riesz, University of Lund, Sweden
R. M. Robinson, University of California
J. B. Rosser, Cornell University
R. Salem, Massachusetts Institute of Technology
I. J. Schoenberg, University of Pennsylvania
M. Shiffman, New York University

P. A. Smith, Columbia University
D. C. Spencer, Stanford University
N. E. Steenrod, University of Michigan
M. H. Stone, University of Chicago
J. L. Synge, Carnegie Institute of Technology
A. Tarski, University of California
T. Y. Thomas, Indiana University
O. Veblen, Institute for Advanced Study
A. Wald, Columbia University
R. J. Walker, Cornell University
J. L. Walsh, Harvard University
J. H. M. Wedderburn, Princeton University
H. Weyl, Institute for Advanced Study
J. H. C. Whitehead, University of Oxford, England
H. Whitney, Harvard University
G. T. Whyburn, University of Virginia
D. V. Widder, Harvard Univesity
N. Wiener, Massachusetts Institute of Technology
R. L. Wilder, University of Michigan
J. W. T. Youngs, Indiana University
O. Zariski, University of Illinois
A. Zygmund, University of Pennsylvania

PROGRAM

PRINCETON UNIVERSITY BICENTENNIAL CONFERENCE

Problems of Mathematics

FIRST DAY - TUESDAY, DECEMBER 17, 1946

Opening of Conference
Chairman: L. P. Eisenhart

Session 1: Algebra
Chairman: E. Artin — Reporter: G. P. Hochschild
Discussion leader(s): G. Birkhoff, R. Brauer, N. Jacobson

Session 2: Algebraic Geometry
Chairman: S. Lefschetz — Reporter: I. S. Cohen
Discussion leader(s): W. V. D. Hodge, O. Zariski

Session 3: Differential Geometry
Chairman: O. Veblen — Reporter: C. B. Allendoerfer
Discussion leader(s): V. Hlavatý, T. Y. Thomas

Session 4: Mathematical Logic
Chairman: A. Church — Reporter: J. C. C. McKinsey
Discussion leader(s): A. Tarski

SECOND DAY - WEDNESDAY, DECEMBER 18, 1946

Session 5: Topology
Chairman: A. W. Tucker — Reporter: S. Eilenberg
Discussion leader(s): H. Hopf, D. Montgomery, N. E. Steenrod,
J. H. C. Whitehead

Session 6: New Fields
Chairman: J. von Neumann — Reporter: V. Bargmann
Discussion leader(s): G. C. Evans, F. D. Murnaghan, J. L. Synge,
N. Wiener

Session 7: Mathematical Probability
Chairman: S. S. Wilks — Reporter: J. W. Tukey
Discussion leader(s): H. Cramér, J. L. Doob, W. Feller

THIRD DAY - THURSDAY, DECEMBER 19, 1946

Session 8: Analysis
Chairman: S. Bochner Reporter: R. P. Boas
Discussion leader(s): L. V. Ahlfors, E. Hille, M. Riesz, A. Zygmund

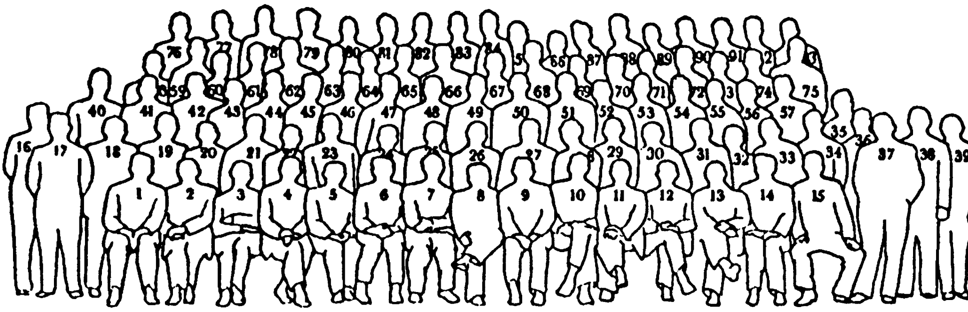
Session 9: Analysis in the Large
Chairman: M. Morse Reporter: M. Shiffman
Discussion leader(s): R. Courant, H. Hopf

Final Dinner
Chairman: H. P. Robertson
Speakers: S. Mac Lane, M. H. Stone, H. Weyl

CONFERENCE COMMITTEE

E. Artin, V. Bargmann, S. Bochner, C. Chevalley, A. Church, R. H. Fox, S. Lefschetz (Chairman), H. P. Robertson, A. W. Tucker, J. W. Tukey, E. P. Wigner, S. S. Wilks

Princeton Bicentennial Conference, 1946



The Problems of Mathematics

1. Morse, M., Institute for Advanced Study
2. Ancocbea, G., University of Salamanaca, Spain
3. Borsuk, K., University of Warsaw, Poland
4. Cramér, H., University of Stockholm, Sweden
5. Hlavaty, V., University of Prague, Czechoslovakia
6. Whitehead, J. H. C., University of Oxford, England
7. Garding, L. J., Princeton
8. Riesz, M., University of Lund, Sweden
9. Lefachetz, S., Princeton
10. Veblen, O., Institute for Advanced Study
11. Hopf, H., Federal Technical School, Switzerland
12. Newman, M. H. A., University of Manchester, England
13. Hodge, W. V. D., Cambridge, England
14. Dirac, P. A. M., Cambridge University, England
15. Hua, L. K., Tsing Hua University, China
16. Tukey, J. W., Princeton
17. Harrold, O. G., Princeton
18. Mayer, W., Institute for Advanced Study
19. Mautner, F. I., Institute for Advanced Study
20. Gödel, K., Institute for Advanced Study
21. Levinson, N., Massachusetts Institute of Technology
22. Cohen, I. S., University of Pennsylvania
23. Seidenberg, A., University of California
24. Kline, J. R., University of Pennsylvania
25. Ellenberg, S., Indiana University
26. Fox, R. H., Princeton
27. Wiener, N., Massachusetts Institute of Technology
28. Rademacher, H., University of Pennnsylvania
29. Salem, R., Massachusetts Institute of Technology
30. Tarski, A., University of California
31. Bargmann, V., Princeton
32. Jacobson, N., The Johns Hopkins University
33. Kac, M., Cornell University
34. Stone, M. H., University of Chicago
35. von Neumann, J., Institute for Advanced Study
36. Hedlund, G. A., University of Virginia
37. Zariski, O., University of Illinois
38. Whyburn, G. T., University of Virginia
39. McShane, E. J., University of Virginia
40. Quine, W. V., Harvard
41. Wilder, R. L., University of Michigan
42. Kaplansky, I., Institute for Advanced Study
43. Bochner, S., Princeton
44. Leibler, R. A., Institute for Advanced Study
45. Hildebrandt, T. H., University of Michigan
46. Evans, G. C., University of California
47. Widder, D. V., Harvard
48. Hotelling, H., University of North Carolina
49. Peck, L. G., Institute for Advanced Study
50. Synge, J. L., Carnegie Institute of Technology
51. Rosser, J. B., Cornell
52. Murnaghan, F. D., The Johns Hopkins University
53. Mac Lane, S., Harvard
54. Cairns, S. S., Syracuse University
55. Brauer, R., University of Toronto, Canada
56. Schoenberg, I. J., University of Pennsylvania
57. Shiffman, M., New York University
58. Milgram, A. N., Institute for Advanced Study
59. Walker, R. J., Cornell
60. Hurewicz, W., Massachusetts Institute of Technology
61. McKinsey, J. C. C., Oklahoma Agricultural and Mechanical
62. Church, A., Princeton
63. Robertson, H. D., Princeton
64. Bullitt, W. M., Bullitt and Middleton, Louisville, Ky.
65. Hille, E., Yale University
66. Albert, A. A., University of Chicago
67. Rado, T., The Ohio State University
68. Whitney, H., Harvard
69. Ahlfors, L. V., Harvard
70. Thomas, T. Y., Indiana University
71. Crosby, D. R., Princeton
72. Weyl, H., Institute for Advanced Study
73. Walsh, J. L., Harvard
74. Dunford, N., Yale
75. Spenser, D. C., Stanford University
76. Montgomery, D., Yale
77. Birkhoff, G., Harvard
78. Kleene, S. C., University of Wisconsin
79. Smith, P. A., Columbia University
80. Youngs, J. W. T., Indiana University
81. Steenrod, N. E., University of Michigan
82. Wilks, S. S., Princeton
83. Boas, R. P., Mathematical Reviews, Brown University
84. Doob, J. L., University of Illinois
85. Feller, W., Cornell University
86. Zygmund, A., University of Pennsylvania
87. Artin, E., Princeton
88. Bohnenblust, H. F., California Institute of Technology
89. Allendoerfer, C. B., Haverford College
90. Robinson, R. M., Princeton
91. Bellman, R., Princeton
92. Begle, E. G., Yale
93. Tucker, A. W., Princeton