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Norbert Wiener and Chaos

BROCKWAY McMILLAN

The roots of Wiener's "Homogeneous Chaos" [W3] can be found in a series of papers from the period 1919–1922, of which "Differential Space" [W1] was the penultimate, and in the later paper "Generalized Harmonic Analysis" [W2] of 1930. The final paper of the 1919–1922 series recasts some of the arguments of [W1] and of the earlier papers. It is [W3] that introduced the term "chaos," a term that did not long retain Wiener's original intended meaning, and is used in a different technical sense today.

The contributions of [W3] over its predecessors were of course more than verbal. They included

(1) the definition of the *pure homogeneous chaos*, an extension to a many dimensional space of the random process on the time axis described in [W1], accomplished by a method different from that of [W1]

(2) a multidimensional ergodic theorem, later sharpened in [W4]

(3) extensions to multidimensional processes of some of the results of [W2] on time series

(4) the definition of the *discrete homogeneous chaos*, a multidimensional random point process, and

(5) a theorem to the effect that a certain class of functionals of the pure chaos is weakly dense within a much larger, but not clearly specified, class of random processes.

The term "chaos," as used by Wiener in [W3] and later, is a noun that, with qualifiers, indicates a kind of random process. A general definition is given in ¶2 of [W3], almost *en passant*: a chaos is a real- or vector-valued function $F(S;\alpha)$. S is drawn from a sufficiently rich class of subsets of n-dimensional Euclidean space E_n, a class later chosen to be a countable ring Ξ that generates the σ-ring of Borel sets. The function $F(S;\alpha)$, as a function of S, is finitely additive over Ξ and for each S is a Lebesgue measurable function of the real variable α on $0 \leq \alpha \leq 1$.

[W3] always makes the "random" label α explicit, and represents the complete probability space, of which α is a representative point, as the unit interval $0 \leq \alpha \leq 1$ with ordinary lebesgue measure.

[W3] does not adhere strictly to the initial definition of "chaos," using the term also to designate a random point function defined on E_n, now often called a random field. By specialization, then, the term also designates any function measurable on $0 \leq \alpha \leq 1$.

Throughout [W1] and its predecessors, there is emphasis on models of natural phenomena, particularly on models of Brownian motion. [W2] focuses on the analysis of observational data in the form of time series, but the element of randomness is not explicitly present. [W3] opens with a clear statement of intent to provide a mathematical basis for the modeling and study of a wide class of random phenomena in nature, mentioning specifically physics and statistical mechanics. Its opening paragraphs display Wiener's deep concern about ergodic theory — the problem of identifying the average of some local physical quantity over an *ensemble*, or random universe of states of a system, with the average of that same quantity over all localities in a particular sample state drawn from that universe. (In Wiener's terms, the average over an ensemble of states is called the "phase average.") This problem of identification is fundamental to the Gibbs approach to statistical mechanics.

Because of the ergodic issue, the paper lays great store by the uniform structure of the underlying Euclidean space E_n and by the invariance, not only of E_n itself but also of the mathematical structures to be erected thereon, under an n-generator group of translations. Hence the "homogeneous" in its title, and the definition, already in ¶2, of homogeneity: $F(\cdot;\cdot)$ is defined to be *homogeneous* if the distributions of the random variables $F(S;\cdot)$, $F(T;\cdot)$ are the same whenever T is a translation of S.

Paragraphs three, four, and five of [W3] deal with ergodic theory, proving an ergodic theorem based on iterates of translations drawn from the symmetry group of E_n, a theorem later sharpened in [W4]. This theorem establishes the existence of the spatial average, over a sample state (i.e., for a chosen α), of a fairly general kind of functional of a chaos. To identify this average with the ensemble average of that functional requires that the chaos be *metrically transitive*: a chaos is metrically transitive if, for each set S, given that $S_{\mathbf{v}}$ is the translation of S by the vector $\mathbf{v}$, the joint distribution of $F(S;\cdot)$ and $F(S_{\mathbf{v}};\cdot)$ tends to independence as $|\mathbf{v}| \to \infty$.

Paragraphs six and seven introduce one of the two random processes of interest in the present essay. In ¶6 we find the one-dimensional *pure homogeneous chaos*, known today as the Wiener process (an interesting synonymy in itself). This is exactly the process considered in [W1], now defined by explicitly introducing a measure into the space of additive set functions on the ring Ξ of subsets of E_1. In the companion papers to [W1], references to an underlying probability measure, in measure-theoretic terms, were absent or at best muted. These papers dealt with function spaces and the averages (expectations) of functionals defined on them. These averages were defined directly as Daniell integrals and the underlying probability measure per se given little explicit attention. [W3] made the probability measure a basic element by mapping elementary events—subsets of the function space called by Wiener *contingencies* and in [Ko] *cylinder sets* — into subsets of the interval $0 \leq \alpha \leq 1$ in such a way that the Lebesgue measure of the image equalled the desired probability of the elementary event.

Between the date, 1921, of [W1] and the publication of [W3], much had happened in the theory of random processes. In particular, Kolmogoroff's fundamental paper [Ko] had appeared. I have no evidence that Wiener ever read [Ko]. Certainly its basic result would have greatly simplified the proofs, in ¶¶6, 7, and 11 of [W3], as well as, later, in [WW], that a stochastic process with specified properties exists. Bochner [B] shows an elegant adaptation of the method of [Ko] specifically to spaces of additive set functions. Bochner told me that this latter paper was inspired by his impatience with Wiener's approach.

[W1] used Brownian motion as its motivation. It dealt with that motion in terms of the paths, or time-histories of displacement, of a particle subjected to random impulses. Accordingly, [W1] introduced a measure into the space of *paths*: functions on $0 \leq t \leq 1$ that vanish at $t = 0$. [W1] showed that this measure assigns outer measure unity to the class of continuous functions. This is now known as the Wiener measure.

A sample path can be thought of as the definite integral, from time zero, of the sample of "white noise" that constitutes the history of buffeting to which the Brownian particle was subject. This is indeed the basis on which [W1] defined the Wiener measure. An important contribution of [W3] then appears in ¶6, for here the point of view is shifted back from *paths* to their *increments*: the Wiener process of ¶6, an additive set function on the one-dimensional time axis, is the *indefinite* integral of a sample of white noise.

This change in point of view is fundamental, for it allows the extension of the theory, in ¶7, to several dimensions.

In ¶7 lies the heart of the paper. The argument of ¶6 is extended from the one-dimensional time axis to E_n and defines the multidimensional Wiener process or *pure homogeneous chaos.* This is a chaos $P(\cdot;\cdot)$ such that for every collection $S_1, S_2, \ldots, S_k$ of pairwise disjoint sets drawn from Ξ the random variables $P(S_1;\cdot), \ldots, P(S_k;\cdot)$ are independent and Gaussian, with means zero and variances respectively equal to $\mu(S_1), \ldots, \mu(S_k)$, where $\mu(\cdot)$ is Lebesgue measure in E_n. That a random process with these properties exists is in fact a theorem, proved from primitive hypotheses. For the proofs in both ¶6 and ¶7 Wiener uses the technique, mentioned above, of mapping measurable events into measurable subsets of the interval $0 \leq \alpha \leq 1$.

The pure chaos $P(\cdot;\alpha)$, as a set function, is neither countably additive nor of bounded variation, but it admits the definition of (stochastic) integrals. Examples are

$$\text{(I1)} \qquad \int f(x)P(dx;\alpha),$$

$$\text{(I2)} \qquad \iint g(x,y)P(dx;\alpha)P(dy;\alpha),$$

integration being over $x \in E_n$, $y \in E_n$. The objects (I1) and (I2) are of course random variables having values, and properties, that justify the suggestive notation. If the integrands here are *simple* functions, step functions measurable on Ξ and taking only finitely many values, the definitions of the corresponding integrals are obvious from the notation. (I1) is then extended from simple functions to integrands $f \in L^2(\mu)$ by continuity in $L^2(d\alpha)$. Multiple integrals such as (I2) are extended from the integrals of simple functions by a convergence in probability $(d\alpha)$.

From the definitions, Wiener shows that the phase average, that is, the expectation, $\int d\alpha$, of (I1) is zero, as is the expectation of any (Im) of odd order m. That of (I2) is given by

$$\int d\alpha \iint g(x,y)P(dx;\alpha)P(dy;\alpha) = \int g(x,x)\mu(dx).$$

More generally, the expectation of (Im) for even values of m is a sum of integrals each of order $m/2$ in μ, in which the integrand appears with its variables identified in pairs in all possible ways. The principle is simple enough but the combinatorics get complicated. One does well so to define his integrands that they vanish when any two variables coincide.

¶9 defines random fields—random functions of a point $z \in E_n$—from integrals such as (I1), (I2) or a general (Im), by replacing the integrands therein with $f(z-x), g(z-x, z-y)$, etc. The discussion then returns in ¶10 to the generalized harmonic analysis of [W2]. By a careful extension of

[W2], it is shown that, except for a null set of labels α, to such a random field can be assigned a (random) multidimensional power spectrum, a nonnegative chaos on frequency space, frequency space being a copy E_n again. For such a random field $G(z;\alpha)$, the power spectrum is the generalized Fourier transform of the sample autocorrelation

$$H(x;\alpha) = \lim V(r)^{-1} \int_r G(x+y;\alpha)G(y;\alpha)\mu(dy),$$

in which "$\int_r$" denotes integration over a sphere about the origin, of radius r and volume $V(r)$, the limit is as $r \to \infty$, and "+" is vector addition in n-space.

The final issue, in regard to the pure chaos, appears in ¶12, identified as the weak approximation theorem. It asserts that a general chaos, as defined in ¶2, can be approximated weakly in distribution by a suitably chosen polynomial functional of the pure chaos. The statement of this theorem is somewhat vague and is inconsistent with its citations of earlier formulas. The subsequent argument is so obscure that I cannot deduce from it exactly what class of chaoses has been proved to admit this approximation in distribution. In fact, a much stronger result now stands. Taking off from the isometry between L^2 on E_n and $L^2(d\alpha)$, defined by the mapping of $f(\cdot)$ on E_n into the function (I1) (on $0 \le \alpha \le 1$), Kakutani [Ka], Ito [It1], and Segal [S] have developed alternative structures within which the results of ¶¶6–10 of [W3] are extended and, in particular, polynomials in the pure chaos appear as a dense set. A large class of random processes on the line are described by measures in function space that are absolutely continuous with respect to the Wiener measure.

¶11 introduces the other chaos of interest here, the *discrete chaos*, or multidimensional Poisson process. The argument starts from primitive assumptions and shows that there exists a random additive set function $D(S;\alpha)$ defined and Lebesgue measurable on $0 \le \alpha \le 1$ for each $S \in \Xi$, such that (i) if S_1 and S_2 are disjoint then the random variables $D(S_1;\cdot)$ and $D(S_2;\cdot)$ are independent and

$$\text{(ii)} \qquad \operatorname{Prob}\{D(S;\cdot) = k\} = \frac{\mu(S)^k}{k!} \exp\{-\mu(S)\}, \qquad k = 0, 1, 2, \ldots .$$

Indeed, it is shown that the assumption (i), and the equality (ii) for $k = 0$, suffice to characterize the chaos $D(\cdot;\cdot)$. The discussion continues, defining the first order stochastic integral $\int f(x)D(dx;\alpha)$ by extension from simple functions $f(\cdot)$ to functions $f \in L^2(\mu)$ in analogy with the case of the pure chaos, and showing that

$$\text{(A)} \qquad \int d\alpha \int f(x)D(dx;\alpha) = \int f(x)\mu(dx).$$

These results, with ¶10, suffice for calculating the power spectrum of the reponse of a resonator (any linear time-invariant dynamical system) to a Poisson time series of unit impulses. The calculation results in a conclusion that has been known, at least in engineering terms, since 1909 [Ca]. It completes the discussion in [W3] of the Poisson process. Our discussion here of that process will resume later.

My first encounter with Norbert Wiener was in an undergraduate class. I had entered MIT, after two years at another school, without full third-year status, and was caught up in a quite irregular program. This put me, in the spring term of 1935, along with a few other misfits, into a hurriedly scheduled class in differential equations with Wiener as our teacher. It seems likely that this was the first undergraduate teaching he had done in some years. The text was that by H. B. Phillips, written for engineers and full of practical-looking problems, problems that could give one a feel for what the terms of a differential equation really meant. It was clear that Wiener enjoyed teaching this material and he taught it in the spirit of the book. He did not try to improve our minds with excessive rigor. With gusto he showed us all the tricks. He tackled the problems with enthusiasm, and brought in new ones of his own — for example, given the tensile strength of steel as then available, how long can you make a suspension bridge? (Answer: not long enough to bridge the Atlantic.) Wiener was always beautifully articulate but, before even a small group, he tended to adopt a somewhat oratorical style. Beyond this, however, his teaching manner was informal, chatty, even avuncular. He enjoyed a question that required a thoughtful answer. I liked him at once.

Differential equations met right after lunch. Wiener usually entered class with a cigarillo in his mouth. He would sneak a few puffs and then put the butt on the chalk rail. Later he would surreptitiously drop the butt into the side pocket of his jacket. We waited all term for him to drop a smoldering butt into that pocket but he never did.

Three years later, in the spring of 1938, it was arranged that I would do a thesis under Wiener, in the field of random processes. The obvious necessary reading of [W1] and [W2] was already under way or accomplished when a hint of something more specific turned up, in the suggestion by Wiener that I look into [Sch] on the shot-effect. Summer passed and the fall term began with two events, equally unforseen and having, to me, comparable immediate impact: New England's first great hurricane and, a few days later, the arrival of the galley proofs of [W3], handed to me by Wiener with the request that I proofread them. I decline to admit responsibility for the many typographical errors and inconsistencies that remain in the published version of [W3], for, by the time I had catalogued the ones I understood, Wiener had already returned the galleys!

Actually, the specific subject matter of [W3] scarcely figured in any discourse between Wiener and me during the roughly fifteen months that I worked under his tutelage. Throughout this period, his consuming interest was to apply the results of [W3] to the modeling of natural phenomena. He saw in the Wiener process (known to him of course as the pure homogeneous chaos) and in the related weak representation theorem a way to represent the distribution-over-states of fluids and fields. Similarly, though the working tools were not as fully developed, he saw in the Poisson process a way to represent the random states of a particulate system, such as a classical molecular gas or fluid, and as a means to model the shot-effect in electronic devices.

Wiener was anxious to get at both of these fields of application. He suggested that for a thesis I develop the calculus of stochastic integrals with respect to the discrete chaos, and of their ensemble averages, telling me to hurry — since "we" had more important problems to work on. As he had known, it was easy to do. I did hurry, and he accepted the result, [Mc1]. The task was easy for two reasons: the discrete chaos $D(\cdot\,;\alpha)$ is nonnegative, and is of bounded variation on Ξ with probability one, facts exploited in [Mc1]. Indeed, more strongly, the process can so be defined on a Euclidean space that with probability one $D(\cdot\,;\alpha)$ is a σ-finite measure on the σ-ring generated by the class of all bounded sets, so that stochastic integrals are unnecessary. This fact was unknown to me then. Whether it was then known to Wiener I am not sure. He never raised the question with me — to answer it would have made a *good* thesis. In hindsight, the fact explains why the Poisson chaos is easy to work with.

During that academic year, and into the summer of 1939, I saw Wiener's work with the Poisson process from the inside, so to speak, as an amanuensis and quasicollaborator. Along with several others, I also had the opportunity to observe from the outside his work with the Wiener process.

Wiener had scheduled for 1938–1939 a repeat of lectures given some years before on Fourier series and integrals. A good set of mimeographed notes was available from the original lectures, prepared by W. T. Martin and others. Exploiting the existence of these notes, Wiener sped through the entire subject matter of the original lectures in but a few weeks. That in itself was someting of an experience for his listeners, but more was to come. He immediately launched into a quick introduction to the Wiener process and then treated us for the remainder of the academic year to a research seminar addressing the problem of modeling or understanding fluid-mechanical turbulence.

None of his listeners were equipped in any way to participate actively in this research effort. It was a one-man show. Twice weekly he would lay before us his latest ideas for an attack on the problem, sometimes covering the blackboard with prodigious and clearly extemporaneous calculations, sometimes simply speculating, sometimes discussing why what he had just been attempting didn't work. The subject matter was interesting, the spectacle was entertaining, but almost invariably he would break off his line of attack, anticipating some mathematical difficulty, long before any difficulty became evident to his mortal listeners. Quite literally, he could see a shock wave ("Verschiebungstoss" was his word for it) coming long before we could. He regularly deplored the lack of a good existence theory for the Navier-Stokes equations.

This line of application did not lead far during Wiener's lifetime. Later developments, as of 1976, are summarized in [DMc]. At a more basic level, the impact of [W3] and of its predecessors on mathematics has been significant. See [It2] for a brief appreciation. [W3] has also, through the works of Kakutant, Ito, and Segal, had some influence on quantum field theory.

Directly upon the appearance of the galley proofs of [W3], Wiener also plunged into an attack on the statistical mechanics of fluids, work in which I was expected to participate. The working tool was of course the Poisson process. It was clear that Wiener found it stimulating to argue with, or perform before, some kind of audience, even a not very responsive one. I was the chosen audience for statistical mechanics, much as the class in Fourier analysis was that for fluid mechanics.

The work on statistical mechanics turned out to be more substantial and more concrete than that on turbulence. It is described in some detail in [DMc]. I was something more than straight man to the Wiener act, serving as amanuensis and working with the intricate and combinatorial calculations that soon dominated the enterprise. I had brief moments of glory before seminars; Wiener regularly left the exposition to me. He even had me write the abstract [WMc] and let me present the nonresults to the AMS meeting in February 1939. Occasionally I had the exquisite pleasure of contributing a computational trick.

The attack on statistical mechanics began with a natural idea: imagine an infinite cloud of points (molecules), a sample from the Poisson process randomly populating the phase space E_6 (three space coordinates, three velocity coordinates). At time $t = 0$ turn on the intermolecular forces and let this cloud evolve according to Newton's laws. In other words, consider the infinite system of differential equations (of first order, since we are in the phase space of one molecule) that governs the state of this cloud. Given a functional of the cloud such as

$$\text{(F2)} \qquad \Psi(f;\alpha) = \sum_x \sum_y f(x,y),$$

these equations will imply a differential equation that describes its growth. Here $f(\cdot,\cdot)$ is, say, a smooth function with bounded carrier such that $f(x,x) \equiv 0$ for all x, the sums are over all x and y in the cloud, and α, as you have guessed, marks the particular cloud at issue. Wiener would call (F2) a polynomial homogeneously of degree two in the chaos from which this particular cloud is a sample. It is in fact a double integral, (a stochastic integral, as of 1939):

$$\Psi(f;\alpha) = \iint f(x,y)D(dx;\alpha)D(dy;\alpha).$$

The differential equation for Ψ allows one to compute its time-derivatives at $t = 0$. The mth such derivative is a polynomial (in $D(\cdot\ ;\cdot)$) such as (F2) is, of degree $m+2$ in $D(\cdot\ ;\cdot)$. It is linear in $f(\cdot,\cdot)$ and its derivatives, and is of degree m in the interparticle potential.

Since the cloud at time $t = 0$ is a sample from the Poisson process, one can calculate the averages ($\int \cdot\, d\alpha$) of the derivatives of Ψ at $t = 0$. These average derivatives describe a formal Maclaurin's series in the time for the average of Ψ at a later time. The key to the whole application of chaos theory is that integrals over α are reducible to integrals over phase space as in the display (A). By integrating these latter by parts to eliminate the derivatives of $f(\cdot,\cdot)$, one can recast the terms as functionals of $f(\cdot,\cdot)$ so that, formally,

$$\text{(B)}\qquad \int \Psi(f;\alpha)d\alpha = \iint f(x,y)\rho_2(x,y;t)\mu(dx)\mu(dy).$$

Here $\rho_2(\cdot,\cdot\ ;t)$ is the second-order density of a point process that describes the distribution-over-states of the cloud at time t, given that it started as a Poisson cloud at time zero. Similar calculations lead to the other densities ρ_m, $m = 2, 3, \ldots$, (ρ_1 is trivial).

Wiener's hope was by this means to derive expansions for the ρ_m as series in powers of such physical parameters as density and temperature. Terms in such series would be sums of multiple integrals involving the interparticle potential. The calculations and results are in fact no less intricate than those found in other approaches to the problem, approaches that were generating an extensive literature at that very time: [MM], [BG] and others. After July 1939, little seems to have been done by Wiener along this line. Later, by quite another method, he did develop formulas for the multiplet-densities ρ_m. The work was submitted for publication, I believe, in the *Journal of Chemical Physics*, but the method and results had been anticipated by a paper already in press [MMo]. Wiener's manuscript has apparently been lost.

Though nothing of significance to statistical mechanics resulted from this work with the Poisson process, two mathematical problems emerged. For a convenient term, define a *true point process*, on Euclidean space E_n, as a complete probability measure on the class of all subsets of E_n that assigns unit probability to a certain subclass Γ. Γ consists of exactly those subsets $\gamma \subset E_n$ such that $\text{card}(\gamma \cap S)$ is finite for every bounded set $S \subset E_n$. Γ can be called the class of *locally finite* sets.

In defining the integral (I1) as a stochastic integral, [W3] explicitly avoids claiming that the Poisson chaos is a true point process. Whether or not it was such a process made no difference to the formal calculations of the work on statistical mechanics but the mathematical question remained open, albeit tacit at the time. Actually, during those calculations it became evident — it is already evident in the display (B) — that the Poisson process itself was irrelevant. What was relevant was some other point process that modeled the distribution-over-states of the physical gas, a distribution that depends of course upon the interparticle potential and depends upon it in a distressingly complicated way.

Display (B) suggests the existence of a distribution-over-states in which pairs near the point $(x, y) \in \mathrm{E}_6 \times \mathrm{E}_6$ occur with density $\rho_2(x, y; t)$. The analog of (B) for functionals Ψ of degree m similarly defines $\rho_m(\cdot, \cdot, \ldots, \cdot\,; t)$ as the density of m-tuplets in E_{6m}. A critical mathematical question then is: does there exist a true point process that exhibits this sequence $\{\rho_m\}$ of densities? This question subsumes that of the nature of the Poisson chaos, because the latter is characterized by a density sequence $\{\lambda^m\}$ in which the constant λ is the intensity of the process (λ is simply a scaling parameter, assumed $= 1$ in (A)).

Wiener and I subsequently and separately worked on these two problems. In the spring of 1940 I communicated to Wiener the fact that a chaos (not claimed to be a true point process) with well-behaved moments exists if and only if a certain set function $\mathrm{E}\{S\}$, definable in terms of the given sequence $\{\rho_m\}$ of densities, is sufficiently regular and has the property of complete monotonicity. (See [Mc3] for complete monotonicity.) The chaos is then characterized by $\mathrm{E}\{\cdot\}$ as a function on Ξ, in that $\mathrm{E}\{S\} = \mathrm{Prob}\{\gamma \cap S = \varnothing\}$. (It is not then an accident that this latter probability alone, in the form of the function $\exp(-\mu(S))$, the archetypical "sufficiently regular" completely monotone function of S (see [Mc2], ¶6.32) sufficed to define the Poisson chaos in ¶11 of [W3].)

Wiener replied to this news by inviting me to join him and Aurel Wintner as co-author of what, after I declined, became [WW]. This latter paper contains a positivity theorem: given the m-tuplet densities, under certain regularity conditions on the putative moments of the process, a chaos exists if all the putative probabilities $\mathrm{Prob}\{\mathrm{card}(\gamma \cap S) = m\}$, $S \in \Xi$, $m = 0, 1, 2, \ldots$, are nonnegative. Here, of course, the putative moments and probabilities are expressed by those formulas in the given densities that would describe them if a chaos did exist. The tool used is the joint factorial-moment generating function, the (putative) expectation of the product $\prod_i (i - z_i)^{\mathrm{card}(\gamma \cap S_i)}$ considered as a function of the complex variables z_i and the pairwise disjoint bounded Borel sets S_i. In fact, it suffices to consider this function only for one variable z and one variable S, and to postulate that it be continuous from above in S, regular for $|1 - z| \leq 1 + \delta(S)$, where $\delta(S) > 0$ for all $S \in \Xi$, and completely monotone in S when $z = 1$ (or for each z in $0 < z < 1$.) This fact was very nearly in the authors' hands.

Though [WW] is weak in not providing criteria for the positivity of the necessary probabilities, it is strong in that it proves, under its own sufficient conditions, that a true point process exists. The proof is not a model of clarity but, basically, it reduces the problem to that within a bounded set T where, since $\gamma \cap T$ is a finite set, measure theory on a compact space is directly available.

A true point process is one example of a set-valued random process, or random set. In 1973 David Kendall, [Ke], proved a definitive theorem on such processes. Given a space X and a sufficiently rich class Σ of subsets of X, [Ke] defines a property of X and Σ that resembles compactness (although X need not have a conventional topology). Given then a set function $\mathrm{E}\{S\}$ defined for $S \in \Sigma$, if $\mathrm{E}\{\cdot\}$ is continuous from above and is completely monotone, then there exists a random set γ such that $\mathrm{E}\{S\} = \mathrm{Prob}\{\gamma \cap S = \varnothing\}$. It is further true that, if Σ is a countable ring that covers X and separates its points, and if the random set is locally finite—i.e., if $\gamma \cap S$ is a finite set with probability one, for $S \in \Sigma$ — then no compactness condition is needed. This fact is confirmed by [WW] and is probably deducible directly from [Ke], but the only proof I have is based on the methods of [Mc2].

Anecdotes about Wiener are of course legion. Many of them are transparent adaptations of known absent-minded-professor jokes and are of doubtful validity. One frequently heard complaints, however, from persons whom Wiener knew well that he would pass them on the street or in the hall, staring straight ahead and giving no sign of recognition. This occasionally happened to me. One explanation cites Wiener's obvious near-sightedness, but this explanation will not hold water, as I can prove from personal experience. During my year of work with him I lived in the then "new" Graduate House, what had been the Riverbank Court Hotel across Massachusetts Avenue from the main MIT building. A regular Saturday afternoon event was a handball game with roommate and fellow student Abraham Schwartz. In inclement weather the comfortable way to the handball court led through the halls of the main building. On several occasions during that year our trot through these corridors was interrupted by a hail from far down the hall: "Oh McMillan... " in Wiener's best theatrical voice. The handball court then waited while we heard Wiener's latest thoughts about the work in progress. In a semidarkened hallway, from perhaps 100 feet away, near-sighted or not, he could recognize me in shorts, sweatshirt, and sneakers (not the typical student's dress in those days).

During 1938–1939 D. J. Struik lectured on the history of mathematics, a late-afternoon two-hour lecture once a week, not for credit. Attendees came from all over. Wiener was one; he always sat in the front row next the window. Regularly, behind him, sat a silent young man named Slutz. After an absence, Slutz reappeared with a handsome beard, a rarity among students in those days. Slutz also abandoned his habitual seat and sat in the center of the front row. Wiener arrived and sauntered across the front of the room toward his usual place. The new beard caught his eye. He sauntered back to the door and turned to survey the competition from a discreet distance. Then he walked briskly up to Slutz, thrust a welcoming hand forward and introduced himself: "My name's Wiener."

I have only pleasant memories of my personal dealings with Norbert Wiener, from the first days of that class in differential equations. He was always cordial and friendly. I sensed a genuine warmth muted by a faintly Continental formality. He was never patronizing nor was there the slightest suggestion of any master–slave relation between professor and student. He always treated me with full respect.

During the school year of our association, there was little opportunity for more than working meetings. During June and July of 1939, however, I lodged in a boarding house near the Wiener summer place in Tamworth, New Hampshire, and we had more contact. Usually I walked the mile to his place for a working session in the morning, returning to my lodgings for lunch, and possibly repeating these trips in the afternoon. Sometimes he would ferry me in his car. There was a period when his family was not in residence and he took his meals with me.

He was a charming and entertaining companion. Strong as his ego was, he almost never talked in a personal vein. I do remember him reminiscing, amused and self-deprecating, about the unsoldierly PFC Wiener, USA, in service at the Aberdeen Proving Ground during World War I. He and I were bridge partners on a few occasions and he played no better than I! He knew a lot about the folkways, history and dialect of the southern New Hampshire region and enjoyed talking about them. One lunchtime he entertained me with thundering recitations of the poetry of Heine. His humor was subtle, seldom bawdy. He liked limericks — we exchanged only clean ones — and loved a bad pun. We enjoyed exploring science-fiction fantasies, distorting a law of physics and exploring the consequences, or inventing simple ways to harness solar energy.

Wiener never gossiped with me. He spoke with respect of the work of von Neumann and urged me to seek von Neumann out when I went to Princeton. He spoke admiringly of J. D. Tamarkin the man, referring to him as "Jacob Davidovitch," and of his work. He went out of his way to express admiration and respect for Solomon Lefschetz and for his work. He never dropped names. I heard nothing from him about his studies in Cambridge or his work in philosophy and the foundations of mathematics. He seemed always to me as his autobiographical works show him: warm, fond of people, more sensitive than he appeared on the surface, and slightly distant from others more from shyness than from arrogance.

It was inspirational, and discouraging, to watch him work. He did not see me as a competitive threat, but the evidence is that he also worked well with young people who might have been seen as threats. His list of successful collaborations with gifted students is fairly long. He worked hard to get me an appointment for the difficult year 1939–1940, and succeeded. I am in his debt for a career well started. I valued his friendship, and I am pleased to acknowledge this debt in a public way.

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