George E. Andrews was awarded a Ph.D. in Mathematics from the University of Pennsylvania in 1964, and has since been on the faculty of Pennsylvania State University, where he became Evan Pugh Professor of Mathematics in 1981. He has also held visiting positions at various institutions worldwide, including MIT, the University of Wisconsin, the University of New South Wales and the University of Strasbourg. His research has involved basic hypergeometric series; partitions; number theory; combinatorics; and special functions.

J. J. Sylvester, Johns Hopkins and Partitions

GEORGE E. ANDREWS[1]

"Those who cannot remember the past are condemned to repeat it."

Santayana

1. Introduction

J. J. Sylvester was one of the great English mathematicians of the nineteenth century. He is probably best known for his work in invariant theory, and a popular biography of Sylvester (along with that of Cayley) appears in Eric Temple Bell's book **[15]** *Men of Mathematics* under the title "Invariant Twins." The title "Invariant Twins" for a biographical sketch of the two is perhaps misleading; for while each was a major force in invariant theory, they were quite distinct as men:

> Cayley was a calm and precise man, born in Surrey and descended from an ancient Yorkshire family. His life ran evenly and successfully. Sylvester, born of orthodox Jewish parents in London, was brilliant, quick-tempered and restless, filled with immense enthusiasms and an insatiable appetite for knowledge. Cayley spent fourteen years at the bar, an experience which, considering where his real talent lay, was time largely wasted. In his

[1] Partially supported by National Science Foundation Grant DMS-8503324.

> youth Sylvester served as professor of mathematics at the University of Virginia. One day a young member of the chivalry whose classroom recitation he had criticized prepared an ambush and fell upon Sylvester with a heavy walking stick. He speared the student with a sword cane; the damage was slight, but the professor found it advisable to leave his post and 'take the earliest possible passage for England.' Sylvester once remarked that Cayley had been much more fortunate than himself: 'that they both lived as bachelors in London, but that Cayley had married and settled down to a quiet and peaceful life at Cambridge; whereas he had never married, and had been fighting the world all his days.' This is a fair summary of their lives.

The above brief summary was given by James R. Newman in *The World of Mathematics* [**21**; p. 340].

Given Sylvester's difficulties at the University of Virginia one might have expected him never to venture across the Atlantic again. However in 1870 Sylvester was forced to retire as "superannuated" from the mathematics professorship at the Royal Military Academy, Woolwich at age 56. He was rather bitter about this and somewhat at loose ends. H. F. Baker [**14**; pp. xxix–xxx] now describes what happened:

> In 1875 the Johns Hopkins University was founded at Baltimore. A letter to Sylvester from the celebrated Joseph Henry, of date 25 August 1875, seems to indicate that Sylvester had expressed at least a willingness to share in forming the tone of the young university; the authorities seem to have felt that a Professor of Mathematics and a Professor of Classics could inaugurate the work of an University without expensive buildings or elaborate apparatus. It was finally agreed that Sylvester should go, securing, besides his travelling expenses, an annual stipend of 5000 dollars "paid in gold." And so, at the age of sixty-one, still full of fire and enthusiasm, ... he again crossed the Atlantic, and did not relinquish the post for eight years, until 1883. It was an experiment in educational method; Sylvester was free to teach whatever he wished in the way he thought best; so far as one can judge from the records, if the object of an University be to light a fire of intellectual interests, it was a triumphant success. His foibles no doubt caused amusement, his faults as a systematic lecturer must have been a sore grief to the students who hoped to carry away note-books of balanced records for future use; but the moral effect of such earnestness ... must have been enormous. His first pupil, his first class, was Professor George Bruce Halsted; he it was who, as recorded in the Commemoration-day Address ... 'would

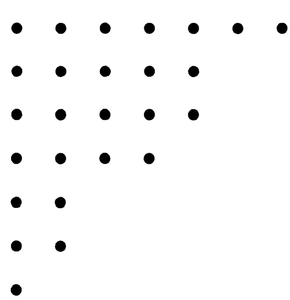
have the New Algebra.' How the consequence was that Sylvester's brain took fire, is recorded in the pages of the American Journal of Mathematics. Others have left records of his influence and methods. Major MacMahon quotes the impressions of Dr. E. W. Davis, Mr. A. S. Hathaway and Dr. W. P. Durfee." (See [**9**; Ch. I, Part A].)

2. Partitions at Johns Hopkins

The modern combinatorial theory of partitions was founded by Sylvester at Johns Hopkins. Most of the work of Sylvester and his students was gathered together in an omnibus paper [**28**] entitled: *A Constructive Theory of Partitions, Arranged in Three Acts, an Interact and an Exodion.* The philosophy of the work is perhaps best summarized in the first paragraph of Act I. On Partitions as Entities [**28**; p. 1]:

> In the new method of partitions it is essential to consider a partition as a definite thing, which end is attained by regularization of the succession of its parts according to some prescribed law. The simplest law for the purpose is that the arrangement of the parts shall be according to their order of magnitude. A leading idea of the method is that of correspondence between different complete systems of partitions regularized in the manner aforesaid. The perception of the correspondence is in many cases greatly facilitated by means of a graphical method of representation, which also serves per se as an instrument of transformation.

A partition of an integer is a representation of it as a sum of positive integers. Thus 7+5+5+4+2+2+1 is a partition of 26. The "graphical method of representation" (or Ferrers graph) referred to associates each summand (or part) m of a partition with a row of m dots. Thus the graph of $7+5+5+4+2+2+1$ is

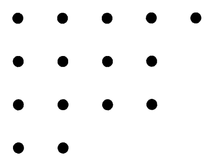


There are numerous interesting results in this 83 page paper. In Sections 3–6 we shall describe how some of the discoveries in Sylvester's magnum opus continue to influence current research.

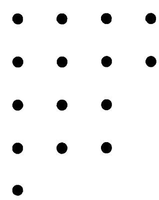
It is my firm belief that Sylvester's paper still deserves study. To emphasize this point, Section 7 is devoted to a new proof of one of Sylvester's identities. Section 8 considers a generalization of this result which leads to a new proof of the Rogers-Ramanujan identities.

3. The Durfee Square Combinatorially

Act II of **[28]** starts off with Durfee's observation that the number of self-conjugate partitions of an integer n equals the number of partitions of n into distinct odd parts. The conjugate of a partition is obtained by interchanging rows and columns in its Ferrers graph. Thus $5+4+4+2$ has graph



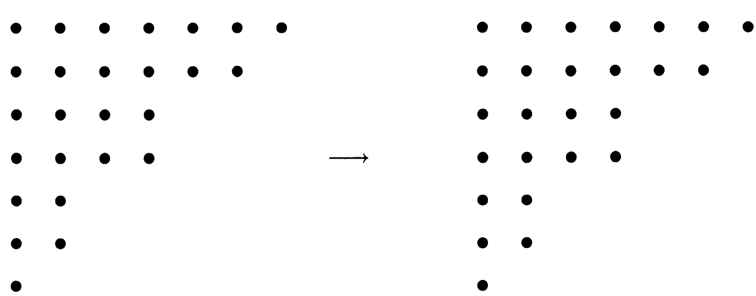
Interchanging rows and columns we get the graph of the conjugate partition



which is the partition $4+4+3+3+1$.

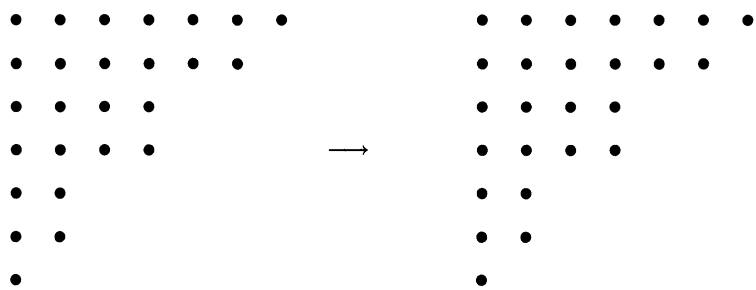
A self-conjugate partition is a partition that is identical with its conjugate.

Let us now return to Durfee's assertion that the self-conjugate partitions of n are equinumerous with the partitions of n into distinct odd parts. A one-to-one correspondence between these two classes of partitions is easily established using the Ferrers graph. To illustrate we start with the self-conjugate partition $7+6+4+4+2+2+1$



Enumerating the nodes in each indicated right angle, we obtain the partition $13+9+3+1$ which has distinct odd parts due to the symmetry of the original graph.

It is then observed that each self-conjugate partition can be dissected into a square and two symmetric parts. For example,



The square of nodes is called the Durfee square, and it is a short graphical argument [**28**; p. 27] to see that the generating function for self-conjugate partitions with Durfee square of side i is

$$\frac{q^{i^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2i})}.$$

Summing over all i, Sylvester determines that in

$$1+\sum_{i=1}^{\infty}\frac{a^i q^{i^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2i})}$$

the coefficient of $a^i q^n$ is the number of partitions of n into self-conjugate partitions with Durfee square of side i. On the other hand, in the infinite product

$$\prod_{i=1}^{\infty}(1+aq^{2i-1})$$

the coefficient of $a^i q^n$ is the number of partitions of n into i distinct odd parts. Hence from Durfee's one-to-one correspondence follows

$$1+\sum_{i=1}^{\infty}\frac{a^i q^{i^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2i})}=\prod_{i=1}^{\infty}(1+aq^{2i-1}), \tag{3.1}$$

an identity of Euler [**7**; p. 19, eq. (2.2.6)].

Much of the remainder of Act II is devoted to a Durfee square analysis of other types of partitions. Next arbitrary partitions of n into i parts are treated and an identity of Cauchy [**28**; p. 30] follows

$$1+\sum_{i=1}^{\infty}\frac{a^i q^{i^2}}{(1-q)(1-q^2)\cdots(1-q^i)(1-aq)(1-aq^2)\cdots(1-aq^i)} = \prod_{i=1}^{\infty}\frac{1}{(1-aq^i)}. \tag{3.2}$$

After (3.2), Sylvester considers partitions of n into i distinct parts and finds [**28**; p. 33]

$$1+\sum_{j=1}^{\infty}\frac{(1+aq)(1+aq^2)\cdots(1+aq^{j-1})(1+aq^{2j})a^j q^{j(3j-1)/2}}{(1-q)(1-q^2)\cdots(1-q^j)} = \prod_{j=1}^{\infty}(1+aq^j). \tag{3.3}$$

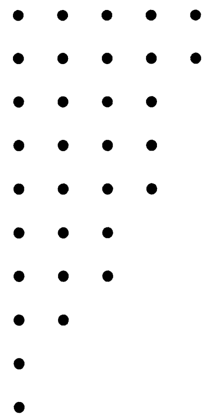
This identity has unexpected benefits, since setting $a=-1$ yields Euler's famous Pentagonal Number Theorem [**7**; p. 11, eq. (1.3.1)]:

$$1+\sum_{j=1}^{\infty}(-1)^j q^{j(3j-1)/2}(1+q^j)=\prod_{j=1}^{\infty}(1-q^j). \tag{3.4}$$

In a related historical study on Sylvester [**8**; Ch. I, Part B], I have examined the interaction of Sylvester's surprising new proof of (3.4) and F. Franklin's famous combinatorial proof of the Pentagonal Number Theorem [**17**]. Indeed Franklin's proof is probably the first major mathematical discovery in American mathematics. It has been presented often before [**7**; pp. 10–11][**28**; p. 11].

Franklin's proof of (3.4) which is a natural outgrowth of Sylvester's analysis of (3.3) has affected research in partitions substantially. I. Schur's [**24**] combinatorial derivation of the Rogers-Ramanujan-Schur identities is clearly built upon Franklin's transformations. In 1981, A. Garsia and S. Milne [**18**] answered the long standing challenge of finding a bijective proof of the Rogers-Ramanujan identities. Their proof relies in an essential manner on Schur's combinatorial work. Thus the Franklin-Sylvester method is clearly in evidence in this major achievement by Garsia and Milne in the 1980s.

Additionally we should mention that the Durfee square analysis has been extended to rectangles in [**4**]; K. Kadell [**20**] has subsequently greatly extended this approach and has been able to derive identities of incredible complexity. M. V. Subbarao [**26**] has generalized (3.4) by noting further invariants in Franklin's transformation, and these observations have been extended to other identities [**5**]. Finally [**9**] Ferrers graphs can be analyzed by putting in successive Durfee squares underneath each other. For example, the Ferrers graph of $5 + 5 + 4 + 4 + 4 + 3 + 3 + 2 + 1 + 1$ is



with five successive Durfee squares. Successive Durfee squares enter into one extension of the Rogers-Ramanujan identities [**9**].

THEOREM. *The number of partitions of n with at most k successive Durfee squares equals the number of partitions of n into parts not congruent to* $0, \pm k \pmod{2k+1}$.

Also successive Durfee squares enter into a general theorem on L. J. Rogers' false theta functions [**8**; Ch. III, Part B].

4. ANALYTIC IMPLICATIONS OF (3.3)

Sylvester's identity (3.3) holds the distinction of being the first q-series identity whose first proof was purely combinatorial. Of this achievement, Sylvester states [**29**; last paragraph]: "Par la même méthode, j'obtiens la série pour les thêta fonctions et d'autres séries beaucoup plus générales, sans calcul algébrique aucun." Thus challenged by Sylvester's remark, A. Cayley [**16**] noted that an elegant and short analytic proof of (3.3) could be given. Namely if $F(a)$ denotes the left-hand side of (3.3), then for $|q| < 1$

(4.1)

$$\begin{aligned} F(a) &= 1 + \sum_{j=1}^{\infty} \frac{(1+aq)(1+aq^2)\cdots(1+aq^{j-1})\{(1-q^j)+q^j(1+aq^j)\}a^j q^{j(3j-1)/2}}{(1-q)(1-q^2)\cdots(1-q^j)} \\ &= 1 + \sum_{j=1}^{\infty} \frac{(1-aq)(1+aq^2)\cdots(1+aq^{j-1})a^j q^{j(3j-1)/2}}{(1-q)(1-q^2)\cdots(1-q^{j-1})} \\ &\quad + \sum_{j=1}^{\infty} \frac{(1+aq)(1+aq^2)\cdots(1+aq^j)a^j q^{j(3j+1)/2}}{(1-q)(1-q^2)\cdots(1-q^j)}. \end{aligned}$$

Now shifting j to $j+1$ in the first sum and combining the two sums, we see that

$$(4.2)\quad F(a) = 1 + aq + \sum_{j=1}^{\infty} \frac{(1+aq)(1+aq^2)\cdots(1+aq^j)a^j q^{j(3j+1)/2}(1+aq^{2j+1})}{(1-q)(1-q^2)\cdots(1-q^j)}$$

$$= (1+aq)F(aq).$$

Once we note that $F(0) = 1$, we can iterate the functional equation (4.2) to derive

$$F(a) = \prod_{j=1}^{\infty}(1+aq^j) \tag{4.3}$$

as desired.

This technique of proof obtained by Cayley to treat Sylvester's identity has been applied over and over again to increasingly complex problems.

The proofs of the Rogers-Ramanujan identities given by Rogers and Ramanujan in **[23]** use exactly Cayley's procedure. A. Selberg **[25]** again uses this method to generalize the Rogers-Ramanujan identities.

The full implications of Cayley's method are developed analytically in **[3]** and are applied to prove the General Rogers-Ramanujan Theorem **[6]**. Actually Rogers understood much more of the generality than he ever published **[10]**. We shall say more about this topic in Section 8.

5. Fishhooks

In Act III of **[28]**, Sylvester gives an ingenious combinatorial proof of a refinement of a famous theorem of Euler:

THEOREM 1. *The number of partitions of an integer n into odd parts equals the number of partitions of n into distinct parts.*

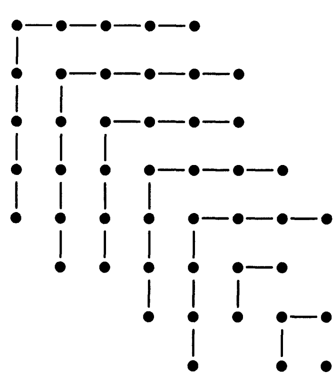
Sylvester **[28**; p. 15] refined this result as follows:

THEOREM 2. *Let $A_k(n)$ denote the number of partitions of n into odd parts (repetitions allowed) with exactly k distinct parts appearing. Let $B_k(n)$ denote the number of partitions of n into distinct parts such that exactly k sequences of consecutive integers occur in each partition. Then $A_k(n) = B_k(n)$.*

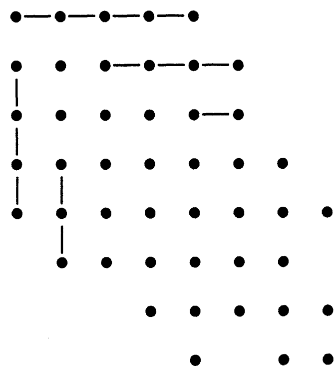
Let us consider $n = 9$ for examples of Euler's theorem and its refinement

	odd parts	distinct parts
$k = 1$	9	$5+4$
	$3+3+3$	$4+3+2$
	$1+1+1+1+1+1+1+1+1$	9
$k = 2$	$7+1+1$	$6+3$
	$5+1+1+1+1$	$7+2$
	$3+3+1+1+1$	$6+2+1$
	$3+1+1+1+1+1+1$	$8+1$
$k = 3$	$5+3+1$	$5+3+1$

Sylvester discovered an ingenious one-to-one mapping between partitions with odd parts and partitions with distinct parts. The mapping is best explained by an example. Let us consider a partition with odd parts say $9+9+7+7+7+3+3+1$. We provide a new graphical representation by a sequence of nested right angles



We now connect these nodes in this graph through a series of 45° angle paths (fishhooks).



The partition thus indicated in $12+10+9+6+4+3+2$.

In fact the partitions of 9 listed above are paired up by the fishhook mapping. The fact that this mapping is a one-to-one mapping takes some care.

This technique has not been as fruitful as the results in Sections 3 and 4; however several authors [**1**], [**12**], [**19**], [**22**] have considered the implications of Sylvester's refinement of Euler's theorem. Also the fishhook method was applied [**2**; p. 136] to give a combinatorial proof of an identity of N. J. Fine.

6. Combinatorics of Jacobi's Triple Product Identity

One of the most important identities in the theory of elliptic theta functions is Jacobi's Triple Product Identity [**7**; p. 21]:

$$\prod_{n=1}^{\infty}(1+zq^n)(1+z^1q^{n-1}) = \frac{\sum_{m=-\infty}^{\infty} z^m q^{\binom{m+1}{2}}}{\prod_{n=1}^{\infty}(1-q^n)}. \tag{6.1}$$

If we denote the left-hand side of (6.1) by $\phi(z)$, then we see that

$$\begin{aligned}\phi(zq) &= \prod_{n=1}^{\infty}(1+zq^{n+1})(1+z^{-1}q^{n-2}) \\ &= z^{-1}q^{-1}\prod_{n=1}^{\infty}(1+zq^n)(1+z^{-1}q^{n-1}) \\ &= z^{-1}q^{-1}\phi(z),\end{aligned} \tag{6.2}$$

and observing that (for $|q|<1$) $\phi(z)$ is analytic in z in a deleted neighborhood of 0, we obtain from (6.2) that

$$\phi(z) = \sum_{m=-\infty}^{\infty} z^m A_m, \tag{6.3}$$

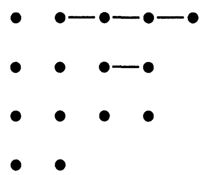
where $A_m = q^m A_{m-1}$.

Consequently by iteration

$$\phi(z) = A_0 \sum_{m=-\infty}^{\infty} z^m q^{(m+1/2)}. \tag{6.4}$$

The Exodion of [28] (partly due to A. S. Hathaway) provides a very nice means of seeing that A_0 is just $\prod_{n\geq 1}(1-q^n)^{-1}$.

Instead of following the Sylvester-Hathaway approach exactly we shall introduce the Frobenius symbol of a partition. To each partition there is associated an equilength two-rowed array of nonnegative integers where each row is strictly decreasing. We illustrate the construction with an example:

$5+4+4+2$:  $\begin{pmatrix} 4 & 2 & 1 \\ 3 & 2 & 0 \end{pmatrix}$.

The procedure for an arbitrary partition is then as follows: In the Ferrers graph, delete the main diagonal then read rows to the right of the diagonal and columns below. In this way we see that each partition can be represented by a Frobenius symbol and vice versa. Note that the length of the rows in the Frobenius symbol is equal to the side of the Durfee square in the partition. Furthermore if $\prod$ is a partition of n and $\binom{a_1\ a_2 \ldots a_r}{b_1\ b_2 \ldots b_r}$ is the related Frobenius symbol, then

$$n = r + \sum_{i=1}^{r}(a_i + b_i).$$

It is now an easy matter to compute A_0 in (6.4). A_0 is the constant term of the left-hand side of (6.1). Contributions to the constant term arise from expressions of the form

$$(zq^{n_1})(zq^{n_2})\cdots(zq^{n_r})(z^{-1}q^{m_1})(z^{-1}q^{m_2})\cdots(z^{-1}q^{m_r})$$

where $n_1 > n_2 > \cdots > n_r \geq 1$ and $m_1 > m_2 > \cdots > m_r \geq 0$, and thus the exponent on q corresponds to the Frobenius symbol

$$\begin{pmatrix} n_1 - 1 & n_2 - 1 & \ldots & n_r - 1 \\ m_1 & m_2 & \ldots & m_r \end{pmatrix}.$$

Hence the constant term A_0 in (6.1) is (as a function of q) the generating function for all Frobenius symbols. But Frobenius symbols correspond to ordinary partitions, and so

$$A_0 = \sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n} \quad ([7;\text{ p. } 4]).$$

This idea is capable of wide generalization and is dealt with at length in an AMS Memoir entitled Generalized Frobenius Partitions **[11]** (see also **[12]**). Recent work by L. Kolitsch and F. Garvan on this topic is surveyed in [**13**; Ch. 7]. It should be noted that this fruitful idea has been rediscovered many times over the years. A history is given in [**11**; p. 4].

7. A Neglected Identity

The discoveries of Sylvester and his students have obviously been quite important in later developments. Thus any such discovery which has not been carefully investigated bears looking into. After proving (3.3), Sylvester devotes a short section [**28**; §36, pp. 33–34] to a finite version of (3.3) and a combinatorial proof thereof. Namely for nonnegative integral i:

$$(-aq)_i = 1 + \sum_{j\geq 1} \begin{bmatrix} i+1-j \\ j \end{bmatrix} (-aq)_{j-1} q^{j(3j-1)/2} a^j + \sum_{j\geq 1} \begin{bmatrix} i-j \\ j \end{bmatrix} (-aq)_{j-1} q^{3j(j+1)/2} a^{j+1}, \tag{7.1}$$

where

$$(A)_j = (A;q)_j = (1-A)(1-Aq)\cdots(1-Aq^{j-1}), \tag{7.2}$$

and

$$\begin{bmatrix} A \\ B \end{bmatrix} = \begin{cases} \dfrac{(1-q^A)(1-q^{A-1})\cdots(1-q^{A-B+1})}{(1-q^B)(1-q^{B-1})\cdots(1-q)}, & 0 \leq B \leq A \\ 0, & \text{otherwise.} \end{cases} \tag{7.3}$$

We propose to give an analytic proof of (7.1).

Let us denote the right-hand side of (7.1) by $S(i;a;q)$ and appropriately apply Cayley's method from Section 4.

$$\begin{aligned} (7.4)\quad & S(i;a;q) \\ &= 1 + \sum_{j\geq 1} \frac{(-aq)_{j-1} q^{j(3j-1)/2} a^j (q)_{i-j}}{(q)_j (q)_{i+1-2j}} \\ &\qquad \times \{(1-q^{i+1-j}) + aq^{2j}(1-q^{i+1-2j})\} \\ &= 1 + \sum_{j\geq 1} \frac{(-aq)_{j-1} q^{j(3j-1)/2} a^j (q)_{i-j}}{(q)_j (q)_{i+1-2j}} \\ &\qquad \times \{(1-q^j) + q^j(1+aq^j)(1-q^{i+1-2j})\} \end{aligned}$$

$$= 1 + \sum_{j\geq 0} \frac{(-aq)_j q^{(j+1)(3j+2)/2} a^{j+1} (q)_{i-j-1}}{(q)_j (q)_{i-1-2j}}$$

$$+ \sum_{j\geq 1} \frac{(-aq)_j q^{j(3j+1)/2} a^j (q)_{i-j}}{(q)_j (q)_{i-2j}}$$

$$= 1 + aq + (1+aq) \left\{ \sum_{j\geq 1} (-aq^2)_{j-1} q^{j(3j-1)/2} (aq)^j \begin{bmatrix} (i-1)+1-j \\ j \end{bmatrix} \right.$$

$$\left. + \sum_{j\geq 1} (-aq^2)_{j-1} q^{j(3j+3)/2} (aq)^{j+1} \begin{bmatrix} i-1-j \\ j \end{bmatrix} \right\}$$

$$= (1+aq) S(i-1; aq; q).$$

We may now iterate (7.4) i times and conclude since $S(0; a; q) = 1$ that

$$S(i; a; q) = (-aq)_i, \tag{7.5}$$

as desired.

8. A New Proof of the Rogers-Ramanujan Identities

Since the developments arising from (3.3) have been substantial, we might expect that (7.1) is, in fact, part of a larger picture. While this is obviously not the place to develop such implications fully, we nonetheless shall demonstrate that (7.1) is just the first instance of an interesting infinite family of polynomials. From this family of polynomials we shall extract a new proof of the Rogers-Ramanujan identities. We define

$$\begin{aligned} &C_{k,h}(i; a; q) \\ &\quad = \sum_{j\geq 0} (aq)_j (-1)^j a^{kj} q^{(1/2)(2k+1)j(j+1)-hj} \begin{bmatrix} i+h-kj \\ j \end{bmatrix} \\ &\quad\quad - \sum_{j\geq 0} (aq)_j (-1)^j a^{kj+h} q^{(1/2)(2k+1)j(j+1)+hj+h} \begin{bmatrix} i-kj \\ j \end{bmatrix}, \end{aligned} \tag{8.1}$$

and we note that

$$\begin{aligned} C_{1,1}(i; a; q) &= (1-aq) S(i; -aq; q) \\ &= S(i+1; -a; q). \end{aligned} \tag{8.2}$$

Also observing term by term cancellation, we see that

$$C_{k,0}(i; a; q) = 0. \tag{8.3}$$

We may again apply Cayley's method to obtain

(8.4)

$$\begin{aligned}
&C_{k,h}(i;a;q) - C_{k,h-1}(i;a;q) \\
&\quad = \sum_{j\geq 0}(aq)_j(-1)^j a^{kj} q^{(1/2)(2k+1)j(j+1)-hj} \\
&\qquad\quad \times \left\{ \begin{bmatrix} i+h-kj \\ j \end{bmatrix} - q^j \begin{bmatrix} i+h-1-kj \\ j \end{bmatrix} \right\} \\
&\qquad + \sum_{j\geq 0}(aq)_j(-1)^j a^{kj+h-1} q^{(1/2)(2k+1)j(j+1)+(h-1)j+h-1} \\
&\qquad\quad \times \begin{bmatrix} i-kj \\ j \end{bmatrix} (1-aq^{j+1}) \\
&\quad = \sum_{j\geq 1}(aq)_j(-1)^j a^{kj} q^{(1/2)(2k+1)j(j+1)-hj} \begin{bmatrix} i+h-1-kj \\ j-1 \end{bmatrix} \\
&\qquad + \sum_{j\geq 0}(aq)_{j+1}(-1)^j a^{kj+h-1} q^{(1/2)(2k+1)j(j+1)+(h-1)j+h-1} \begin{bmatrix} i-kj \\ j \end{bmatrix} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by [7; p. 35, eq. (3.3.4)])} \\
&\quad = \sum_{j\geq 0}(aq)_{j+1}(-1)^{k(j+1)} q^{(1/2)(2k+1)(j+1)(j+2)-h(j+1)} \begin{bmatrix} i+h-1-k-kj \\ j \end{bmatrix} \\
&\qquad + \sum_{j\geq 0}(aq)_{j+1}(-1)^j a^{kj+h-1} q^{(1/2)(2k+1)j(j+1)+(h-1)j+h-1} \begin{bmatrix} i-kj \\ j \end{bmatrix} \\
&\quad = a^{h-1}q^{h-1}(1-aq) \\
&\qquad\quad \times \left\{ \sum_{j\geq 0}(aq^2)_j(-1)^j(aq)^{kj} q^{(1/2)(2k+1)j(j+1)-(k-h+1)j} \begin{bmatrix} i-kj \\ j \end{bmatrix} \right. \\
&\qquad - \sum_{j\geq 0}(aq^2)_j(-1)^j(aq)^{kj+k-h+1} q^{(1/2)(2k+1)j(j+1)+(k-h+1)j+k-h+1} \\
&\qquad\quad \left. \times \begin{bmatrix} i-(k-h+1)-kj \\ j \end{bmatrix} \right\} \\
&\quad = a^{h-1}q^{h-1}(1-aq)C_{k,k-h+1}(i-(k-h+1);aq;q).
\end{aligned}$$

To prepare for the Rogers-Ramanujan identities we consider (8.4) for $k = 2, h = 1$ and 2.

$$C_{2,2}(i;a;q) - C_{2,1}(i;a;q) = aq(1-aq)C_{2,1}(i-1;aq;q), \tag{8.5}$$

$$C_{2,1}(i;a;q) = (1-aq)C_{2,2}(i-2;aq;q). \tag{8.6}$$

Eliminating $C_{2,1}(i;a;q)$ we find

$$\begin{aligned} C_{2,2}(i;a;q) &= (1-aq)C_{2,2}(i-2;aq;q) \\ &\quad + aq(1-aq)(1-aq^2)C_{2,2}(i-3;aq^2;q). \end{aligned} \tag{8.7}$$

We shall also require some related polynomials

$$D(i;a;q) = \sum_{0\le 2j\le i} \begin{bmatrix} i-j \\ j \end{bmatrix} a^j q^{j^2}. \tag{8.8}$$

$$\Delta(i;a;q) = C_{2,2}(i;a;q) - (aq)_{[i/2]+1} D\left(\left[\frac{i}{2}\right]+2;a;q\right). \tag{8.9}$$

We note the first few values of these polynomials in the following short table.

i	$C_{2,2}(i;a;q)$	$(aq)_{[i/2]+1}D\left(\left[\frac{i}{2}\right]+2;a;q\right)$	$\Delta(i;a;q)$
0	$1-a^2q^2$	$(1-aq)(1+aq)$	0
1	$1-a^2q^2-a^2q^3+a^3q^4$	$(1-aq)(1+aq)$	$-a^2q^3+a^3q^4$
2	$1-a^2q^2-a^2q^3-a^2q^4$ $+a^3q^4+a^3q^5$	$(1-aq)(1-aq^2)$ $\times(1+aq+aq^2)$	0
3	$1-a^2q^2-a^2q^3-a^2q^4$ $-a^2q^5+a^3q^4+a^3q^5+a^3q^6$ $+a^4q^9-a^5q^{10}$	$(1-aq)(1-aq^2)$ $\times(1+aq+aq^2)$	$-a^2q^5+a^3q^6$ $+a^4q^9-a^5q^{10}$

From this table (and extensions computed by means of IBM's SCRATCHPAD) we conjecture the following result.

THEOREM 3. *$a^{-2}q^{-2i-1}\Delta(2i-1;a;q)$ and $a^{-3}q^{-2i-4}\Delta(2i;a;q)$ are polynomials in a and q.*

PROOF. These assertions are immediate from our table for $\Delta(i;a;q)$ with $i\le 3$. To begin with we note that

$$\begin{aligned} D(i;a;q) &= \sum_{j\ge 0} \begin{bmatrix} i-j \\ j \end{bmatrix} a^j q^{j^2} \\ &= \sum_{j\ge 0} \left(\begin{bmatrix} i-j-1 \\ j-1 \end{bmatrix} + q^j \begin{bmatrix} i-j-1 \\ j \end{bmatrix} \right) a^j q^{j^2} \\ &\qquad \text{(by [7; p. 35, eq. (3.3.4)])} \\ &= D(i-1;aq;q) + \sum_{j\ge 0} \begin{bmatrix} i-j-2 \\ j \end{bmatrix} a^{j+1} q^{(j+1)^2} \\ &= D(i-1;aq;q) + aqD(i-2;aq^2;q). \end{aligned} \tag{8.10}$$

Additionally we require

$$
\begin{aligned}
&(1-aq^i)D(i+1;a;q)-D(i;a;q)\\
&=\sum_{j\geq 0}\left(\begin{bmatrix} i+1-j \\ j\end{bmatrix} a^j q^{j^2} - \begin{bmatrix} i-j \\ j\end{bmatrix} a^j q^{j^2}\right)\\
&\qquad -aq^i\sum_{j\geq 0}\begin{bmatrix} i+1-j \\ j\end{bmatrix} a^j q^{j^2}\\
&=\sum_{j\geq 1}\begin{bmatrix} i-j \\ j-1\end{bmatrix} a^j q^{i+(j-1)^2} - aq^i\sum_{j\geq 0}\begin{bmatrix} i+1-j \\ j\end{bmatrix} a^j q^{j^2}\\
&=aq^i\sum_{j\geq 0}\left(\begin{bmatrix} i-j-1 \\ j\end{bmatrix} - \begin{bmatrix} i+1-j \\ j\end{bmatrix}\right) a^j q^{j^2}\\
&=aq^i\sum_{j\geq 0}\left(-q^{i+1-2j}\begin{bmatrix} i-j \\ j-1\end{bmatrix} - q^{i-2j}\begin{bmatrix} i-j-1 \\ j-1\end{bmatrix}\right) a^j q^{j^2}\\
&\qquad\qquad\qquad\qquad ([7;\text{ p. 35, eq. (3.3.3) twice}])\\
&=-a^2q^{2i}D(i-1;a;q)-a^2q^{2i+1}D(i-2;a;q).
\end{aligned}
\tag{8.11}
$$

Let us now assume that the assertions of the Theorem are true for each subscript $< 2i$. Then

$$
\begin{aligned}
\Delta(2i;a;q) &= C_{2,2}(2i;a;q)-(aq)_{i+1}D(i+2;a;q)\\
&=(1-aq)C_{2,2}(2(i-1);aq;q)\\
&\qquad +aq(1-aq)(1-aq^2)C_{2,2}(2(i-1)-1;aq^2;q)\\
&\qquad -(aq)_{i+1}(D(i+1;aq;q)+aqD(i;aq^2;q)\\
&=(1-aq)\Delta(2(i-1);aq;q)\\
&\qquad +aq(1-aq)(1-aq^2)\Delta(2(i-1)-1;aq^2;q).
\end{aligned}
\tag{8.12}
$$

Multiplying by $a^{-3}q^{-2i-4}$ we obtain

(8.13)

$$
\begin{aligned}
&a^{-3}q^{-2i-4}\Delta(2i;a;q)\\
&\quad =q(1-aq)(aq)^{-3}q^{-2(i-1)-4}\Delta(2(i-1);aq;q)\\
&\qquad +(1-aq)(1-aq^2)(aq^2)^{-2}q^{-2(i-1)-1}\Delta(2(i-1)-1;aq^2;q).
\end{aligned}
$$

Thus by our induction hypothesis applied to the indices $2i-2$ and $2i-3$ we see that $a^{-3}q^{-2i-4}\Delta(2i;a;q)$ is a polynomial in a and q. Next

$$
\begin{aligned}
\Delta(2i+1;a;q) &= C_{2,2}(2i+1;a;q) - (aq)_{i+1}D(i+2;a;q) \\
&= (1-aq)C_{2,2}(2i-1;aq;q) \\
&\quad + aq(1-aq)(1-aq^2)C_{2,2}(2i-2;aq^2;q) \\
&\quad - (aq)_{i+1}(D(i+1;aq;q) + aqD(i;aq^2;q)) \\
&= (1-aq)\Delta(2i-1;aq;q) \\
&\quad + aq(1-aq)(1-aq^2)\Delta(2i-2;aq^2;q) \\
&\quad + aq(aq)_{i+2}D(i+1;aq^2;q) \\
&\quad - aq(aq)_{i+1}D(i;aq^2;q) \\
&= (1-aq)\Delta(2i-1;aq;q) \\
&\quad + aq(1-aq)(1-aq^2)\Delta(2i-2;aq^2;q) \\
&\quad + aq(aq)_{i+1}(-a^2q^{2i+4}D(i-1;aq^2;q) \\
&\quad - a^2q^{2i+5}D(i-2;aq^2;q)).
\end{aligned} \tag{8.14}
$$

Multiplying by $a^{-2}q^{-2i-3}$, we obtain

$$
\begin{aligned}
&a^{-2}q^{-2i-3}\Delta(2i+1;a;q) \\
&\quad = (1-aq)(aq)^{-2}q^{-2i-1}\Delta(2i-1;aq;q) \\
&\qquad + a^2q^6(1-aq)(1-aq^2)(aq^2)^{-3}q^{-2i-2}\Delta(2i-2;aq^2;q) \\
&\qquad - aq^2(aq)_{i+1}D(i-1;aq^2;q) \\
&\qquad - aq^3(aq)_{i+1}D(i-2;aq^2;q).
\end{aligned} \tag{8.15}
$$

Thus each expression on the right-hand side of (8.15) is either obviously a polynomial in a and q or is such due to the induction hypothesis.

This completes the induction and our theorem is valid. □

COROLLARY. (*The Rogers-Ramanujan Identities* [7; p. 104]).

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}, \tag{8.16}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}. \tag{8.17}$$

PROOF. We note immediately that the left-hand side of (8.16) is $D(\infty;1;q)$, and Jacobi's Triple Product [7; p. 22, Cor. 2.9] shows that the

$C_{2,2}(\infty; 1; q)/(q)_\infty$ is the right-hand side of (8.16). Furthermore inspection of (8.1) and (8.8) shows that the coefficient of $a^r q^n$ in either $D(i; a; q)$ or $C_{2,2}(i; a; q)$ is fixed for $i \geq i_0(r, n)$. Hence for any r and n the coefficient of $a^r q^n$ in

$$C_{2,2}(\infty; a; q) - (aq)_\infty D(\infty; a; q)$$

must be zero since it is zero in

$$C_{2,2}(m; a; q) - (aq)_{[m/2]+1} D\left(\left[\frac{m}{2}\right] + 1; a; q\right)$$

for $m \geq n - 1$ by Theorem 3. Consequently

$$C_{2,2}(\infty; a; q) = (aq)_\infty D(\infty; a; q), \tag{8.18}$$

and (8.18) reduces to (8.16) when $a = 1$.

By (8.6) and (8.18),

$$\begin{aligned} C_{2,1}(\infty; a; q) &= (1 - aq) C_{2,2}(\infty; aq; q) \\ &= (aq)_\infty D(\infty; aq; q), \end{aligned} \tag{8.19}$$

and (8.19) reduces to (8.17) when $a = 1$. □

9. Conclusion

In this article we have tried to show the importance of Sylvester's contributions to partitions during his tenure at Johns Hopkins. We have not given a full account of all his work on partitions; indeed, he gave a lengthy series of lectures **[27]** on the asymptotics of partitions earlier in his career. Furthermore we have omitted some of the topics (for example those connected with Farey series) of **[28]** which seem less directly related to later work on partitions.

As is abundantly obvious, Sylvester's work was the first truly serious combinatorial study of partitions, and his ideas are still relevant in today's research. Whether the comments in Sections 7 and 8 have more than passing interest remains to be seen. In any event, study of the discoveries of this brilliant mathematician "full of fire and enthusiasm" is still quite worthwhile.

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