AN EXISTENCE THEOREM OF HARMONIC FUNCTIONS
WITH POLYNOMIAL GROWTH

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Abstract. We prove the existence of nonconstant harmonic functions with polynomial growth on manifolds with nonnegative Ricci curvature, Euclidean volume growth and unique tangent cone at infinity.

INTRODUCTION

For a noncompact, complete Riemannian manifold \((M^n, p)\) with nonnegative Ricci curvature,

\[(0.1) \quad \text{Ric}_{M^n} \geq 0,\]

we have the notion of tangent cone at infinity, which is any pointed Gromov-Hausdorff limit of some sequence \(M_i = (M^n, R_i^{-2}dx^2)\) with \(R_i \to \infty\).

The almost rigidity theorem of Cheeger and Colding \cite{4} implies that if \(M^n\) has Euclidean volume growth, i.e., there is some \(V_1 > 0\) such that for all \(R > 0\),

\[(0.2) \quad \text{Vol}(B_R(p)) \geq V_1 R^n,\]

then every tangent cone at infinity is a metric cone, i.e., \(\mathbb{R}_+ \times X\) with the metric \(dr^2 + r^2dx^2\); here \((X, dx^2)\) is a metric space with diameter no more than \(\pi\).

In this paper we will prove

Theorem 0.1. Assume that \(M^n\) is a complete Riemannian manifold satisfying \((0.1)\) and \((0.2)\). Assume that \(M\) has a unique tangent cone \(C(X)\) at infinity. Then the dimension of the space of harmonic functions on \(M^n\) with

\[(0.3) \quad |u(y)| \leq C(1 + d(p, y)^N)\]

is at least \(C(V_\infty)N^{n-1}\); here \(C(V_\infty) > 0\).

For each \(N > 0\), the space of harmonic functions \(u\) with \((0.3)\) on manifolds with \((0.1)\) is finite dimensional; this was conjectured by Yau and proved by Colding and Minicozzi in \cite{11}. See, for example, \cite{12, 16} for further developments.

The tangent cone at infinity may not be unique; see \cite{14, 5}. However, it is unique if we assume that the sectional curvature is nonnegative. Moreover, the example of Menguy \cite{18} shows that even if \(M^n\) has unique tangent cone, \(M^n\) can have infinite topological type.

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By inspecting the proof of Theorem 0.1 we have, when the tangent cones are not unique,

**Theorem 0.2.** Assume that $M^n$ is a complete Riemannian manifold satisfying (0.1) and (0.2). Assume there exists $\lambda > 0$, such that for all tangent cones $C(X)$, $\lambda$ is greater than $\lambda_1(X)$, the first eigenvalue of the Laplacian on $X$, and $\lambda$ is not an eigenvalue of any $X$. Then there exists a nonconstant polynomial growth harmonic function on $M^n$.

It seems the example in [5] satisfies the assumption above and so admits a nonconstant polynomial growth harmonic function.

There are manifolds that do not admit nonconstant harmonic functions with polynomial growth. For example, the manifold obtained by rounding off the end of $\mathbb{R}^+ \times S^{n-1}$; one can check this directly or by [20]. Note this example satisfies (0.1) but not (0.2).

In [13], the author showed that there is a separation of variables formula for the Laplacian on $C(X)$. In particular, there exist many harmonic functions on $C(X)$. We will transplant these harmonic functions back to balls on $M^n$: we then construct the desired harmonic functions by the Arzela-Ascoli theorem. In order to control the growth of these functions, we use a monotonicity Lemma 1.2, which is a generalization of the monotonicity of frequency for harmonic functions on $\mathbb{R}^n$ (see [1], [10], [9]).

Suppose that $(M_i^n, \text{Vol}_i) \xrightarrow{dGH} (M_\infty, \mu_\infty)$ in the measured Gromov-Hausdorff sense, i.e., the sequence $\{M_i^n\}$ converges in the Gromov-Hausdorff sense to $M_\infty$, and for any $x_i \to x_\infty$ ($x_i \in M_i^n$) and $R > 0$, we have $\text{Vol}_i(B_R(x_i)) \to \mu_\infty(B_R(x_\infty))$. In fact, for any sequences of manifolds with Ricci curvature bounded from below, after possible renormalization of the measures when $\{M_i^n\}$ is collapsing, there is a subsequence that converges in the measured Gromov-Hausdorff sense; moreover, under assumption (0.2), $\mu_\infty$ is just the $n$-Hausdorff measure on $M_\infty$. See [5].

**Definition 0.3.** Suppose $K_i \subset M_i^n$ $\xrightarrow{dGH} K_\infty \subset M_\infty$ in the measured Gromov-Hausdorff sense. $f_i$ is a function on $M_i^n, i = 1, 2, \ldots$; $f_\infty$ is a continuous function on $M_\infty$. Assume that $\Phi_i : K_\infty \to K_i$ are $\epsilon_i$-Gromov-Hausdorff approximations, $\epsilon_i \to 0$. If $f_i \circ \Phi_i$ converge to $f_\infty$ uniformly, we say that $f_i \to f_\infty$ uniformly over $K_i$ $\xrightarrow{dGH} K_\infty$.

For a Lipschitz function $f$ on $M_\infty$, one can define a norm

$$\|f\|^2_{H^1_2} = \|f\|^2_{L^2} + \int_{M_\infty} |\text{Lip } f|^2,$$

where $\text{Lip } f$ is the pointwise Lipschitz constant

$$\text{Lip } f(x) = \lim_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)}.$$

In [3], a Sobolev space $H^1_2$ is constructed by taking the closure of the norm (0.4). Moreover, one can define the differential $df$ for $H^1_2$ functions $f$. In [3] it is proved that $M_\infty$ is $\mu_\infty$-rectifiable, and, as a corollary, (0.4) comes from an inner product $\langle \cdot, \cdot \rangle$. Thus $H^1_2$ transforms to a Hilbert space. Now by the standard theory of Dirichlet forms, one gets a positive self-adjoint Laplacian $\Delta$ on $M_\infty$,

$$\int_{M_\infty} \langle df, dg \rangle = \int_{M_\infty} (\Delta f)g;$$
see Theorem 6.25 of [6].

The general philosophy is that the Laplacian $\Delta_i$ over $M_i$ "converge" to the operator $\Delta$ on $M_\infty$. We have the persistence of Poisson’s equation [3], [6], [14]:

**Lemma 0.4.** Assume that $\Delta u_i = f_i$ on (a subset of) $M_i$, $\text{Lip} u_i, \text{Lip} f_i \leq L$ for some $L > 0$. Assume that $u_i \to u_\infty, f_i \to f_\infty$ uniformly. Then on $M_\infty$ we have $\Delta u_\infty = f_\infty$.

We use some standard notation. Write

\[(0.7)\]

\[\int_W f = \frac{1}{\text{Vol}(W)} \int_W f.\]

Denote by $A(p, R_1, R_2)$ the metric annulus \{x| $R_1 \leq d(p, x) \leq R_2$\}. For any function $u_i$ we denote by $u_{i,p,R}$ the average of $u_i$ over $A(p, R/2, R)$:

\[(0.8)\]

\[u_{i,p,R} = \frac{1}{A(p, R/2, R)} \int_{A(p, R/2, R)} u_i.\]

The Laplacian operators are assumed to be positive.

After finishing this manuscript, Professor Colding pointed out to the author a paper of Zhang [22], in which nonconstant harmonic functions of polynomial growth can be constructed in the case when $C(X)$ is a smooth cone. Our construction turns out to be a generalization of [22] and applies to the case when $C(X)$ is not a smooth cone (so there are no coordinate systems available).

1. Analysis on metric cones

It is easy to see (13) that the $(n-1)$-Hausdorff measure on the cross section $X$ satisfies a doubling condition and the Poincare inequality. Moreover, the rectifiability as in [6] holds on $X$ as well; so one can define a Laplacian $\Delta_X$ on $X$. We have an eigenfunction expansion \{\phi_i\} with $\Delta_X \phi_i = \lambda_i \phi_i$ on $X$. By the standard Moser iteration, the $\phi_i$ are H"older continuous; later we will see that they are Lipschitz.

On a metric cone $C(X)$, there is a separation of variable formula [13]:

\[(1.1)\]

\[\Delta u = -\frac{\partial^2 u}{\partial r^2} - \frac{n-1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \Delta_X u.\]

Therefore, if $\phi_i$ is the $i$-th eigenfunction of $\Delta_X$ on $X$ with eigenvalue $\lambda_i$, then $r^{\alpha_i} \phi_i(x)$ is harmonic; here $\alpha_i$ is the unique positive number with

\[(1.2)\]

\[\lambda_i = \alpha_i (n + \alpha_i - 2).\]

We normalize so that $\|\phi_i\|_{L^2(X)} = 1$. Assume $u$ is harmonic on $B_2(p) \subset C(X)$. Then we can write (see [2], [8])

\[(1.3)\]

\[u = \sum_{i=0}^{\infty} c_i r^{\alpha_i} \phi_i.\]

Define

\[(1.4)\]

\[I(r) = \frac{1}{\text{Vol}(B_r(p_\infty))} \int_{\partial B_r(p_\infty)} u^2;\]

here Vol is the $(n-1)$-Hausdorff measure; see [5]. $p_\infty$ is the pole of $C(X)$. Then

\[(1.5)\]

\[I(r) = \sum_{i=0}^{\infty} c_i^2 r^{2\alpha_i}.\]
Similarly to the Euclidean case \([14]\), we have

**Lemma 1.1.** There is a \(k > 1\) that depends only on \(X\) such that for \(\epsilon > 0\) sufficiently small, if \(u\) is harmonic, then

\[
I(r) \leq (2^{\alpha_1+\epsilon})^2 I(r/2) \tag{1.12}
\]

implies

\[
I(r/2) < (2^{\alpha_1+\epsilon})^2 I(r/4). \tag{1.17}
\]

**Proof.** By (1.5), (1.6) is equivalent to

\[
\sum_{\alpha \neq \alpha_1} c_i^2 r^{2\alpha_i} (1 - \frac{2^{\alpha_1+2\epsilon}}{2^{\alpha_i}}) \leq \sum_{\alpha_i=\alpha_1} c_i^2 r^{2\alpha_i} (2^{2\epsilon} - 1). \tag{1.18}
\]

On the other hand, (1.7) is equivalent to

\[
\sum_{\alpha \neq \alpha_1} \frac{1}{2^{\alpha_i}} c_i^2 r^{2\alpha_i} (1 - \frac{2^{\alpha_1+2\epsilon}}{2^{\alpha_i}}) \leq \sum_{\alpha_i=\alpha_1} \frac{1}{2^{\alpha_i}} c_i^2 r^{2\alpha_i} (2^{2\epsilon} - 1). \tag{1.19}
\]

Thus, it suffices to show for \(\alpha_1 \neq \alpha_1\),

\[
\frac{1}{2^{\alpha_i}} (2^{\alpha_1} - 2^{\alpha_1+\epsilon})/(2^{\alpha_1} - 2^{\alpha_1+2\epsilon}) < \frac{1}{2^{\alpha_i}} (2^{2\epsilon} - 1)/(2^{2\epsilon} - 1). \tag{1.10}
\]

Since there is a definite gap (that depends on \(X\)) between \(\alpha_1\) and those \(\alpha_i \neq \alpha_1\), the above holds when \(k > 1\) is sufficiently close to 1 and \(\epsilon\) sufficiently small. \(\Box\)

**Corollary 1.11.** Assume \(u\) is harmonic. If

\[
\int_{A(p_\infty, r/2, r)} u^2 \leq (2^{\alpha_1+\epsilon})^2 \int_{A(p_\infty, r/4, r/2)} u^2, \tag{1.12}
\]

then

\[
\int_{A(p_\infty, r/4, r/2)} u^2 < (2^{\alpha_1+\epsilon})^2 \int_{A(p_\infty, r/8, r/4)} u^2. \tag{1.13}
\]

**Lemma 1.2.** For \(\epsilon\) small enough (as in Corollary \([1, 11]\)), there exist \(\delta, H > 0, k > 1\) depending only on \(\epsilon\) such that if a manifold \((M, p)\) satisfies \([0, 1]\),

\[
d_{GH}(B_4(p), B_4(p_\infty)) < \delta \tag{1.14}
\]

then for any harmonic function \(u\) over \(B_2(p)\), the inequality

\[
\int_{A(p, 1/2, 1)} |u - u_{p, 1/2}|^2 \leq (2^{\alpha_1+\epsilon})^2 \int_{A(p, 1/4, 1/2)} |u - u_{p, 1/2}|^2 \tag{1.15}
\]

implies

\[
\int_{A(p, 1/4, 1/2)} |u - u_{p, 1/2}|^2 < (2^{\alpha_1+\epsilon})^2 \int_{A(p, 1/8, 1/4)} |u - u_{p, 1/4}|^2. \tag{1.16}
\]

**Proof.** The proof is similar to the arguments in \([14]\). Assume the lemma is not true; then for \(\delta_j \to 0\), we can find a sequence of harmonic functions \(u_i\) that satisfies (1.13) but not (1.10). After suitable renormalization, by the Cheng-Yau gradient estimate, a subsequence of \(u_i\) will converge to a function \(u_\infty\) on \(C(X)\) satisfying (1.13) but not (1.10). Now by Lemma (1.4) \(u_\infty\) is harmonic, so we get a contradiction to Corollary (1.11). \(\Box\)
**Lemma 1.3.** For all $\epsilon$ small enough, there exists $\delta$ such that if a manifold $(M,p)$ satisfies (0.1) and (0.2), and

\begin{equation}
(1.17) \quad d_{GH}(B_2(p), B_2(p_\infty)) < \delta
\end{equation}

$(B_2(p_\infty) \subset C(X))$, then for any nonconstant harmonic function $u$ over $B_2(p)$,

\begin{equation}
(1.18) \quad \int_{A(p,1/2,1)} |u - u_{p,1}|^2 \geq (2^{n-1} - \epsilon)^2 \int_{A(p,1/4,1/2)} |u - u_{p,1/2}|^2.
\end{equation}

**Proof.** This is clearly true for harmonic functions on the metric cone $C(X)$. The proof follows from a compactness argument like the previous lemma. \qed

Similarly, we have

**Lemma 1.4.** For $\epsilon < 1$, there exist $\delta > 0, k > 1$ such that if a manifold $(M,p)$ satisfies (0.1) and (0.2), and

\begin{equation}
(1.19) \quad d_{GH}(B_4(p), B_4(p_\infty)) < \delta
\end{equation}

$(B_2(p_\infty) \subset C(X))$, then for any harmonic function $u$ over $B_2(p)$, the inequality

\begin{equation}
(1.20) \quad \left| \int_{A(p,1,2)} u \right| \leq \epsilon \left( \int_{A(p,1,2)} |u|^2 \right)^{\frac{1}{2}}
\end{equation}

implies

\begin{equation}
(1.21) \quad \left| \int_{A(p,2,4)} u \right| \leq \frac{\epsilon}{k} \left( \int_{A(p,2,4)} |u|^2 \right)^{\frac{1}{2}}.
\end{equation}

2. The Barrier and Applications

**Theorem 2.1.** Assume $u_\infty$ is harmonic on the closed ball $B_R(p) \subset C(X)$. Then $u_\infty$ is the uniform limit of a sequence of harmonic functions $u_i$ on $B_R(p_i) \subset M_i$.

**Proof.** We approximate $u_\infty|_{\partial B_R(p_\infty)}$ by Lipschitz functions, then by the transplantation theorem of Cheeger (Lemma 10.7 of [1]) we transplant it back to $M_i$ to a Lipschitz function $\beta_i$ on $\partial B_R(p_i) \subset M_i$,

\begin{equation}
(2.1) \quad \beta_i \to u_\infty|_{\partial B_R(p_\infty)}.
\end{equation}

Solve the Dirichlet problem

\begin{equation}
(2.2) \quad \begin{cases} \Delta u_i &= 0, \\ u_i &= \beta_i &\text{on } \partial B_R(p_i). \end{cases}
\end{equation}

Since $M_i \overset{d_{GH}}{\to} C(X)$, when $i$ is getting bigger we see the ball $B_R(p_i)$ almost satisfies an exterior sphere condition; see [15].

Fix $X_\infty \in \partial B_R(p_\infty)$. Pick $x_i \in \partial B_R(p_i)$ with $x_i \to x_\infty$. On the cone $C(X)$ there is a unique ray starting from the pole $p_\infty$, passing through $x_\infty$. Pick a point $q_\infty$ on this ray with $d(p_\infty, q_\infty) > d(p_\infty, x_\infty)$. Pick $q_i \in M_i$ with $q_i \to q_\infty$.

Consider $b_i(x) = d(q_i, x)^{2-n} - d(q_i, x)^{2-n}$. By the Laplacian comparison theorem,

\begin{equation}
(2.3) \quad \Delta b_i \leq 0.
\end{equation}
Thus exactly as in Chapter 2 of [10] we get two side bounds of $u_i$ near the boundary. Precisely, for all $\epsilon > 0$ there exists $\delta$ such that for $x_i \in \partial B_R(p_i)$, $d(x, x_i) \leq \delta$ implies $|u_i(x) - u_i(x_i)| \leq \epsilon$, when $i$ is sufficiently large.

Now by the Arzela-Ascoli theorem, (a subsequence of) $u_i$ converges to some limit function $v_\infty$ on $C(X)$. By our estimate near the boundary and the maximum principle on $C(X)$, [9], $v_\infty = u_\infty$. \hfill \Box

Note our argument does not imply that $u_i$ is continuous at the boundary.

By the Cheng-Yau gradient estimate we have

Corollary 2.4. Harmonic functions on $C(X)$ are Lipschitz. The eigenfunctions $\phi_i$ on $X$ are Lipschitz.

Corollary 2.5. The first eigenvalue $\lambda_1$ of $\Delta_X$ on $X$ satisfies $\lambda_1 \geq n - 1$.

Proof. The first eigenvalue $\lambda$ gives a harmonic function $r^{\alpha_1}\phi_1(x)$ on $C(X)$. Since it is Lipschitz, $\alpha_1 \geq 1$. By (1.2) we have $\lambda_1 \geq n - 1$. \hfill \Box

This is a generalization of the Lichnerowicz theorem. However, the Obata theorem does not hold: any $X$ such that $C(X)$ splits off some $R$ satisfies $\lambda_1 = n - 1$.

3. Proof of Theorem 0.1

We now prove Theorem 0.1. Pick any sequence $R_i \to \infty$.

By the almost rigidity theorem of Cheeger-Colding [4], there exists a critical radius $R_c$ for $\alpha_1$ such that for all $r > R_c$, the assumptions of Lemma 1.2 Lemma 1.3 and Lemma 1.4, i.e., (1.1), (1.2), (1.4), hold on the rescaled manifold $(M^n, r^{-2}dx^2)$.

As in the previous section we transplant $u_\infty = r^n\phi_1(x)$ back to harmonic functions $u_i$ on $B_2(p_i) \subset M_i = (M^n, R_i^{-2}dx^2)$ so that $u_i \to u_\infty$ uniformly.

We scale back and view $u_i$ as functions on $M^n$. By Theorem 2.1 for $R_i$ sufficiently large, at scale $R_i$ the harmonic function $u_i$ is close to some function $u_\infty = cr^n\phi_1(x)$. Here and below, close means $L^\infty$-close, after an obvious rescale.

So, in particular, we can apply the monotonicity Lemma 1.2 in fact, we iterate it until the scale of critical radius $R_c$ when (the rescaled version of) (1.4) fails. So for all $R$ with $R_c \leq R \leq R_i$,

$$\int_{A(p, R/2, R)} |u_i - u_{i, p, R}|^2 \leq (2^{\alpha_1 + r})^2 \int_{A(p, R^2/4, R/2)} |u_i - u_{i, p, R/2}|^2; \quad (3.1)$$

here recall $u_{i, p, R}$ is the average of $u_i$ on $A(p, R/2, R)$.

Clearly $u_i$ is not a constant. We first subtract a constant and then multiply by a constant so that we can assume

$$\int_{A(p, R_c/2, R_c)} u_i = 0, \quad \int_{A(p, R_c/2, R_c)} u_i^2 = 1. \quad (3.2)$$

So by iterating Lemma 1.4 for all $R$ with $R_c \leq R \leq R_i$,

$$|u_{i, p, R}| = \left| \int_{A(p, R, R)} u_i \right| \leq \epsilon \left( \int_{A(p, R, R)} u_i^2 \right)^{1/2}. \quad (3.3)$$

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We have
\begin{equation}
\left( \int_{A(p,R_c,2R_c)} u_i^2 \right)^{1/2} \leq \left( \int_{A(p,R_c,2R_c)} |u_i - u_{i,2R_c}|^2 \right)^{1/2} + |u_{i,2R_c}|
\leq 2^{\alpha_1+\epsilon} \left( \int_{A(p,R_c/2,R_c)} u_i^2 \right)^{1/2} + \epsilon 2^{\alpha_1+\epsilon} \left( \int_{A(p,R_c/2,R_c)} u_i^2 \right)^{1/2}
\leq 2^{\alpha_1+2\epsilon} \left( \int_{A(p,R_c/2,R_c)} u_i^2 \right)^{1/2}.
\end{equation}

Iterating this, we have
\begin{equation}
\left( \int_{A(p,2^{j-1}R_c,2^jR_c)} u_i^2 \right)^{1/2} \leq 2^{(\alpha_1+2\epsilon)j} \left( \int_{A(p,R_c/2,R_c)} u_i^2 \right)^{1/2}.
\end{equation}

So \( u_i \) (defined on \( B_{R_i}(p) \), with \( R_i \gg R_c \)) is of polynomial growth,
\begin{equation}
|u_i| \leq C \epsilon^{\alpha_1+2\epsilon}.
\end{equation}

Combining with the Cheng-Yau gradient and the Arzela-Ascoli theorem, \( u_i \) converges to a nonconstant polynomial growth harmonic function \( u^{(1)} \) on \( M \).

Next, we indicate how to construct a second harmonic function when there is another eigenfunction for \( \lambda_1 \). By construction, \( u^{(1)} \) satisfies (3.1) and (3.3) at every scale \( R > R_c \). So by Lemma 1.4 on any sufficiently large scale, \( u^{(1)} \) is close to a function of the form
\begin{equation}
\sum_{\alpha_1=\alpha_1} c_{i_1} R^{\alpha_1} \phi_i(x)
\end{equation}
on \( C(X) \). Note that we have no control over the constants \( c_i \). By assumption, \( \lambda_1 \) has more than one multiple; so there is a function of the form
\begin{equation}
\sum_{\alpha_1=\alpha_1} b_{i_1} R^{\alpha_1} \phi_i(x)
\end{equation}
that is perpendicular to (3.7) on \( C(X) \). Like the construction of \( u^{(1)} \), we transplant (3.8) back to \( M_i \), solve the Dirichlet problem as in (2.2), and get a sequence of harmonic functions \( u_i^{(2)} \). Now adjust \( u_i^{(2)} \) by a tiny constant, then subtract \( cu^{(1)} \), a multiple of our first harmonic function \( u^{(1)} \), so that
\begin{equation}
u_i^{(2)} := (u_i^{(2)} - cu^{(1)}) \perp u^{(1)} \text{ on } A(p,R_c,2R_c).
\end{equation}

Note that we have no control over the constant \( c \), but this is not important since all we need is that on scale \( R_i \) we have the inequality (3.1), and \( u_i^{(2)} \) is not a constant. Then as before we construct our second function \( u^{(2)} \). It is independent of \( u^{(1)} \) since it is perpendicular to \( u^{(1)} \) on \( A(p,R_c,2R_c) \).

The constructions of all the other harmonic functions follow the same pattern. Note then we need a revised version of Lemma 1.2 in which \( \alpha_1 \) is substituted by \( \alpha_i \). The generalization is straightforward.

Clearly, if we have \( N \) eigenvalues of \( X \) with \( \lambda \leq \Lambda = N(N+n-2) \), then we have at least \( N \) independent nonconstant harmonic functions \( u^{(j)} \) with
\begin{equation}
|u^{(j)}(y)| \leq C(j,\epsilon)(1+d(p,y)^{N+\epsilon}).
\end{equation}
Now we can count them. By a well-known argument in estimating upper bounds of eigenvalues (similar to p. 105 of [21]), we have

\[(3.11) \quad \lambda_j \leq C(n) \left( \frac{j}{H^{n-1}(X)} \right)^\frac{n}{n-1}; \]

here $H^{n-1}(X)$ is the $(n-1)$-Hausdorff measure of $X$. Actually, we can take $V_\infty$ in (1.2) for it; see [5]. So there are at least $C(V_\infty) \Lambda$ many eigenvalues less than $\Lambda$, and the dimension of harmonic functions with

\[(3.12) \quad |u(y)| \leq C(1 + d(p, y)^N)\]

is at least $C(V_\infty)N^{n-1}$.

Finally, we remark that the technical assumption in Theorem 0.2 is needed to guarantee that Lemma 1.2 works when $C(X)$ is not unique.

REFERENCES


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