AN APPLICATION OF SCHAUDER’S FIXED POINT THEOREM WITH RESPECT TO HIGHER ORDER BVPS

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ABSTRACT. We shall provide conditions on the function $f(t, u_1, \cdots, u_{n-1})$.

The higher order boundary value problem

(BVP)

\[
\begin{cases}
(E) & u^{(n)}(t) + f(t, u(t), u^{(1)}(t), \cdots, u^{(n-2)}(t)) = 0 \text{ for } t \in (0, 1) \text{ and } n \geq 2, \\
(BC) & u^{(i)}(0) = 0, \ 0 \leq i \leq n - 3, \\
& \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\
& \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0
\end{cases}
\]

has at least one solution.

1. Introduction

In this article, we shall attempt to construct some existence criteria for the following $n$-th order boundary value problem:

(BVP)

\[
\begin{cases}
(E) & u^{(n)}(t) + f(t, u(t), u^{(1)}(t), \cdots, u^{(n-2)}(t)) = 0 \text{ for } t \in (0, 1) \text{ and } n \geq 2, \\
(BC) & u^{(i)}(0) = 0, \ 0 \leq i \leq n - 3, \\
& \alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\
& \gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0
\end{cases}
\]

The motivation for the present work stems from many recent investigations in [1]–[3], [8], [13], [23]–[24]. In fact, particular cases of the boundary value problem (BVP) occur in various physical phenomena [4]–[7], [9]–[10], [13], especially such as gas diffusion through porous media, thermal self-ignition of a chemically active mixture of gases in a vessel [7], catalysis theory [9], chemically reacting systems, as well as adiabatic tubular reactor processes. For other related works, we refer to recent contributions of Agarwai and Wong [1]–[3], Anuradaha, Hai and Shivaji [4], Bailey, Shampine and Waltman [5], Erbe and Wang [13], Granas, Guenther and Lee [16], Lee and O’Regan [21], Chyan and Henderson [8], Henderson [17], Vasilev and Klokov [23] and Kelevedjiev [18]–[19] and the references therein.

Here, we shall remark that there are four main techniques to treat the existence of (BVP) as follows:

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Method 01. Shooting method ([25]). This method has been used successfully in the study of some special boundary value problems, if one can guarantee the uniqueness of related initial value problems.

Method 02. Fixed point index method ([1]–[2], [11], [20]). This method has many advantages in treating non-singular boundary value problems and relies on the following lemma:

Let $E$ be a Banach space, and let $C \subseteq E$ be a cone in $E$. Assume that $\Omega_1, \Omega_2$ are open subsets of $E$ with $0 \in \Omega_1, \Omega_1 \subset \Omega_2$, and let

$$T : C \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow C$$

be a completely continuous operator such that either

(i) $||Tu|| \leq ||u||$, $u \in C \cap \partial \Omega_1$, and $||Tu|| \geq ||u||$, $u \in C \cap \partial \Omega_2$; or

(ii) $||Tu|| \geq ||u||$, $u \in C \cap \partial \Omega_1$, and $||Tu|| \leq ||u||$, $u \in C \cap \partial \Omega_2$.

Then $T$ has a fixed point in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Method 03. Nonlinear alternative or topological method ([6], [15]–[16], [22]). This method was initiated in Granas, Guenther and Lee [15]–[16]:

Let $X, Z$ be real vector normed spaces, $L : \text{dom}L \subset X \rightarrow Z$ a linear Fredholm mapping of index zero, $\Omega \subset X$ an open bounded subset, and $N : \overline{\Omega} \rightarrow Z$ an $L$-compact mapping. If $\ker L = \{0\}$, $0 \in \Omega$ and

$$Lx - \muNx \neq 0$$

for every $(x, \mu) \in (\text{dom}L \cap \partial \Omega) \times (0, 1)$, then the equation

$$Lx = Nx$$

has at least one solution in $\text{dom}L \cap \overline{\Omega}$.

Method 04. Schauder’s or Barrier’s method ([12]). In the next section, we attempt to establish a general existence principle for (BVP), which relies on Schauder’s fixed point theorem:

Let $C$ be a convex subset of a normed linear space $E$. Then every compact continuous function $T : C \longrightarrow C$ has at least one fixed point.

2. Main results

Let $\alpha, \gamma, \beta, \delta \geq 0$, $\rho := \gamma \beta + \alpha \gamma + \alpha \delta > 0$ and $B$ be the Banach space

$$\{ u \in C^{(n)}(0, 1) \cap C^{(n-1)}[0, 1] \mid u^{(i)}(0) = 0, \ 0 \leq i \leq n-3 \}$$

with norm $||u|| = \sup_{t \in [0,1]} |u^{(n-2)}(t)|$.

In order to abbreviate our discussion, we suppose throughout this paper that the following assumptions hold:

$(C_1)$: $K(t, s)$ is the Green’s function of the differential equation

$$-u^{(n)}(t) = 0 \text{ in } (0, 1)$$

subject to the boundary conditions $(BC)$.

$(C_2)$: $k(t, s)$ is the Green’s function of the differential equation

$$-u''(t) = 0 \text{ in } (0, 1)$$
subject to the boundary conditions

(BC^*) \begin{cases} \alpha u(0) - \beta u'(0) = 0, \\ \gamma u(1) + \delta u'(1) = 0. \end{cases}

(C_3): f \in C([0, 1] \times \mathbb{R}^{n-1}; \mathbb{R}).

(C_4): v, w \in \mathcal{B} are lower solutions and upper solutions of (BVP) in the sense:

\[ \begin{cases} (1^0) \quad v^{(n)}(t) + f(t, v(t), v^{(1)}(t), \ldots, v^{(n-2)}(t)) \geq 0 \quad \text{for} \quad t \in (0, 1), \\ (2^0) \quad w^{(n)}(t) + f(t, w(t), w^{(1)}(t), \ldots, w^{(n-2)}(t)) \leq 0 \quad \text{for} \quad t \in (0, 1), \\ (3^0) \quad v^{(i)}(0) = 0, \quad 0 \leq i \leq n - 3, \\ (4^0) \quad \alpha v^{(n-2)}(0) - \beta v^{(n-1)}(0) \leq 0, \\ \gamma v^{(n-2)}(1) + \delta v^{(n-1)}(1) \leq 0, \\ (3^0) \quad w^{(i)}(0) = 0, \quad 0 \leq i \leq n - 3, \\ (4^0) \quad \alpha w^{(n-2)}(0) - \beta w^{(n-1)}(0) \geq 0, \\ \gamma w^{(n-2)}(1) + \delta w^{(n-1)}(1) \geq 0, \end{cases} \]

respectively.

(C_5): f(t, u_1, \cdots, u_{n-2}, u_{n-1}), v(t) and w(t) satisfy

\[ v^{(n-2)}(t) \leq w^{(n-2)}(t) \quad \text{on} \quad [0, 1], \quad \text{and} \]

\[ f(t, v(t), \cdots, v^{(n-3)}(t), u_{n-1}) \leq f(t, u_1, \cdots, u_{n-2}, u_{n-1}) \leq f(t, w(t), \cdots, w^{(n-3)}(t), u_{n-1}) \]

for \( t \in [0, 1], \) \( (v(t), \cdots, v^{(n-3)}(t)) \leq (u_1, \cdots, u_{n-2}) \leq (w(t), \cdots, w^{(n-3)}(t)), \)

in which

\[ (x_1, \cdots, x_{n-2}) \leq (y_1, \cdots, y_{n-2}) \iff x_i \leq y_i \quad \text{for} \quad i = 1, \cdots, n - 2. \]

Remark 2.1. It is clear that

(a) if \( f(t, u_1, \cdots, u_{n-2}, u_{n-1}) \) is increasing with respect to \( (u_1, \cdots, u_{n-2}) \) on \( \mathbb{R}^{n-2} \) for each fixed \( t, u_{n-1} \in [0, 1] \times \mathbb{R}, \) then \( C_5 \) holds;

(b) a simple calculation can show that

\[
\frac{\partial^{n-2}}{\partial t^{n-2}} K(t, s) = k(t, s)
\]

\[
= \begin{cases} \frac{1}{\rho} (\beta + \alpha s) \{ \delta + \gamma (1 - t) \}, & 0 \leq s \leq t \leq 1, \\ \frac{1}{\rho} (\beta + \alpha t) \{ \delta + \gamma (1 - s) \}, & 0 \leq t \leq s \leq 1; \end{cases}
\]

(c) there exists an \( M \in (0, 1) \) such that

\[
\begin{cases} (R_1) \quad \frac{k(t,s)}{k(s,s)} \leq 1, & \text{for } t \in [0, 1] \text{ and } s \in [0, 1], \\ (R_2) \quad \frac{k(t,s)}{k(s,s)} \geq M, & \text{for } t \in [\frac{1}{4}, \frac{3}{4}] \text{ and } s \in [0, 1]. \end{cases}
\]

Now, we can state and prove our main result:

**Theorem 2.2** (Main result). **Boundary value problem (BVP) has at least one solution** \( u \in \mathcal{B} \) **such that**

\[ v^{(i)}(t) \leq u^{(i)}(t) \leq w^{(i)}(t) \quad \text{on} \quad [0, 1] \quad \text{for} \quad i = 0, 1, \cdots, n - 2. \]
Proof. We separate the proof into the following steps:
Step (1). Consider the modified problem
\begin{align*}
\begin{cases}
\left(E^*\right) & u^{(n)}(t) + f^*(t, u(t), u^{(1)}(t), \cdots, u^{(n-2)}(t)) = 0 \quad \text{for } t \in (0, 1), \\
(BC) & \begin{cases}
u^{(i)}(0) = 0, & 0 \leq i \leq n - 3, \\
\alpha u^{(n-2)}(0) - \beta u^{(n-1)}(0) = 0, \\
\gamma u^{(n-2)}(1) + \delta u^{(n-1)}(1) = 0,
\end{cases}
\end{cases}
\end{align*}
where

\begin{align*}
f^*(t, u_1, \cdots, u_{n-1}) & := f(t, \eta_1, \cdots, \eta_{n-1}) + \rho(\eta_{n-1} - u_{n-1}), \\
\eta_i & := \begin{cases}
w^{(i-1)}(t) & \text{if } u_i > w^{(i-1)}(t), \\
u_i & \text{if } v^{(i-1)}(t) \leq u_i \leq w^{(i-1)}(t), \\
v^{(i-1)}(t) & \text{if } u_i < v^{(i-1)}(t),
\end{cases}
\end{align*}

for all $i = 1, 2, \cdots, n - 1$, $t \in [0, 1]$ and $\rho : \mathbb{R} \to [-1, 1]$ is the radial retraction defined by

\[\rho(r) := \begin{cases}
r & \text{for } |r| \leq 1, \\
\frac{r}{|r|} & \text{for } |r| > 1.
\end{cases}\]

It is clear that \((BVP^*)\) has a solution $u = u(t)$ if, and only if, $u$ is the solution of the operator equation

\[u(t) = \int_0^1 K(t, s) f^*(s, u(s), u^{(1)}(s), \cdots, u^{(n-2)}(s)) ds := (T u)(t), \quad u \in \mathbb{B},\]
or

\[u^{(n-2)}(t) = \int_0^1 k(t, s) f^*(s, u(s), u^{(1)}(s), \cdots, u^{(n-2)}(s)) ds := (T u^{(n-2)})(t), \quad u \in \mathbb{B}.
\]

Since $f^*$ is continuous and bounded on $[0, 1] \times \mathbb{R}^{n-1}$, $T : \mathbb{B} \to \mathbb{B}$ is continuous and compact. Therefore, it follows from Schauder's fixed point theorem (cf. Method 04) that $T$ has a fixed point $u \in \mathbb{B}$, i.e. \((BVP^*)\) has a solution $u \in \mathbb{B}$.

Step (2). Let

\[H(t) := u^{(n-2)}(t) - w^{(n-2)}(t) \quad \text{on } [0, 1].\]

Then we have

\[H''(\theta) \geq -f^*(\theta, u(\theta), \cdots, u^{(n-3)}(\theta), u^{(n-2)}(\theta)) + f(\theta, w(\theta), \cdots, w^{(n-3)}(\theta), w^{(n-2)}(\theta)) = -f(\theta, \eta_1, \cdots, \eta_{n-2}, w^{(n-2)}(\theta)) - \rho(w^{(n-2)}(\theta) - u^{(n-2)}(\theta)) + f(\theta, w(\theta), \cdots, w^{(n-3)}(\theta), w^{(n-2)}(\theta)) (\eta_i \text{ is defined as in Step (1)}) \geq -\rho(w^{(n-2)}(\theta) - u^{(n-2)}(\theta)) > 0 \quad \text{by using } (C_5)\]

if $\theta \in (0, 1)$ such that $H(\theta) > 0$.

Therefore, we see that there is no $\theta \in (0, 1)$ such that $H(\theta) > 0$ and $H''(\theta) \leq 0$. 

Step (3). Now, we claim that $H(t) \leq 0$ on $[0, 1]$. Suppose to the contrary that there exists a $t_0 \in [0, 1]$ such that $H(t_0) > 0$. Then there is a $\theta \in [0, 1]$ such that

$$H(\theta) := \max_{t \in [0, 1]} H(t) > 0.$$  

Case(1). Suppose that $\beta = 0$, which implies $H(0) \leq 0$ and $\theta \in [0, 1]$.

(1) Suppose that $\delta = 0$, which implies $H(1) \leq 0$ and $\theta \in [0, 1)$. Thus, we have $\theta \in [0, 1)$ and $H''(\theta) \leq 0$. This contradicts the conclusion of Step (2).

(2) Suppose that $\gamma = 0$, which implies $H'(1) \leq 0$. It is clear that $\theta = 1$. In fact, if $\theta \in (0, 1)$, then $H''(\theta) \leq 0$. This gives a contradiction.

Since $H(\theta) = H(1) > 0$, there exists an $\epsilon > 0$ such that $H(t) > 0$ in $(1 - \epsilon, 1]$. Thus $H''(t) > 0$ in $(1 - \epsilon, 1)$, which implies $H'(t)$ is strictly increasing on $[1 - \epsilon, 1]$. It follows from $H'(t) < H'(1) = H'(\theta) = 0$ on $[1 - \epsilon, 1]$ and we see that $H(1) = H(\theta)$ cannot be the maximum of $H(t)$. This gives a contradiction.

Case(2). Suppose that $\alpha = 0$, which implies $H'(0) \geq 0$.

(4) Suppose that $\delta = 0$, which implies $H(1) \leq 0$ and $\theta \in [0, 1)$. It is clear that $\theta = 0$. In fact, if $\theta \in (0, 1)$, then $H''(\theta) \leq 0$. This gives a contradiction.

Suppose that $H'(\theta) = H'(0) > 0$: then $H(t)$ is strictly increasing near $t = \theta = 0$. This implies $H(\theta) = H(0)$ cannot be the maximum of $H(t)$, thus we obtain $H'(0) = 0$.

Since $H(\theta) = H(0) > 0$, there exists an $\epsilon > 0$ such that $H(t) > 0$ in $[0, \epsilon)$. Thus $H''(t) > 0$ in $(0, \epsilon)$, which implies $H'(t)$ is strictly increasing on $[0, \epsilon]$. It follows from $H'(t) > H'(0) = H'(\theta) = 0$ on $[0, \epsilon]$ and we see that $H(0) = H(\theta)$ cannot be the maximum of $H(t)$. This gives a contradiction.

Case(3). Suppose that $\gamma = 0$, which implies $H'(1) \leq 0$. By Case(1)-(2) and Case(2)-(4), we see that $H(0) > 0$ and $H(1) > 0$.

By continuity and Step (2), we see that there exist $t_1$, $t_2 \in (0, 1)$ such that

$$t_1 < t_2, \ H(t) > 0, \ H''(t) > 0 \text{ on } (0, t_1) \cup (t_2, 1).$$

It follows from $H'(0) \geq 0$ and $H'(1) \leq 0$ that

$$H'(t) > 0 \text{ on } (0, t_1) \text{ and } H'(t) < 0 \text{ on } (t_2, 1).$$
This implies that there exists a $\theta \in (t_1, t_2)$ such that
\[ H(\theta) > 0, \quad H'(\theta) = 0 \quad \text{and} \quad H''(\theta) \leq 0, \]
which contradicts the conclusion of Step (2).

(6.0) Suppose that $\gamma \delta > 0$, which implies
\[ w^{(n-2)}(1) \geq \frac{-w^{(n-1)}(1) \delta}{\gamma} \quad \text{and} \quad u^{(n-2)}(1) = \frac{-u^{(n-1)}(1) \delta}{\gamma}. \]
By Case(2)-(5.0), we see that
\[ u^{(n-1)}(1) > w^{(n-1)}(1). \]
Hence, we have
\[ w^{(n-2)}(1) \geq \frac{-w^{(n-1)}(1) \delta}{\gamma} > \frac{-w^{(n-1)}(1) \delta}{\gamma} = u^{(n-2)}(1). \]
It follows from Case(2)-(4.0) that we obtain a contradiction.

Case(3). Suppose that $\delta = 0$, which implies
\[ H'(1) \leq 0. \]
By Case(1)-(2.0) and Case(2)-(5.0), we see that
\[ \alpha \beta > 0 \quad \text{and} \quad u^{(n-2)}(0) > w^{(n-2)}(0). \]
Thus, we have
\[ w^{(n-1)}(0) \leq \frac{u^{(n-2)}(0) \alpha}{\beta} < \frac{u^{(n-2)}(0) \alpha}{\beta} = w^{(n-1)}(0). \]
It follows from Case(2)-(4.0) that we obtain a contradiction.

Case(4). Suppose that $\gamma = 0$, which implies
\[ H'(1) \leq 0. \]
By Case(1)-(2.0) and Case(2)-(5.0), we see that
\[ \alpha \beta > 0 \quad \text{and} \quad u^{(n-1)}(0) < w^{(n-1)}(0). \]
Thus, we have
\[ w^{(n-2)}(0) \geq \frac{u^{(n-1)}(0) \beta}{\alpha} > \frac{u^{(n-1)}(0) \beta}{\alpha} = w^{(n-2)}(0). \]
It follows from Case(1)-(2.0) that we obtain a contradiction.

Case(5). Suppose that $\alpha \beta \gamma \delta > 0$. By Case(1)-(3.0), Case(2)-(6.0), Case(3) and Case(4), we see that
\[ u^{(n-2)}(0) > w^{(n-2)}(0), \quad u^{(n-1)}(0) < w^{(n-1)}(0), \]
\[ u^{(n-2)}(1) > w^{(n-2)}(1), \quad u^{(n-1)}(1) > w^{(n-1)}(1). \]
Thus, we have
\[ w^{(n-1)}(0) \leq \frac{w^{(n-2)}(0) \alpha}{\beta} < \frac{u^{(n-2)}(0) \alpha}{\beta} = u^{(n-1)}(0) < w^{(n-1)}(0), \]
which gives a contradiction.

From Cases(1)-(5), we see that
\[ u^{(n-2)}(t) \leq w^{(n-2)}(t) \quad \text{on} \quad [0, 1]. \]
Similarly, we may show that
\[ u^{(n-2)}(t) \leq w^{(n-2)}(t) \quad \text{on} \quad [0, 1]. \]
Since $u, \ v, \ w \in \mathbb{B}$ and satisfy
\[ u^{(n-2)}(t) \leq u^{(n-2)}(t) \leq u^{(n-2)}(t) \quad \text{on} \quad [0, 1], \]
we obtain
\[ u^{(i)}(t) \leq u^{(i)}(t) \leq u^{(i)}(t) \quad \text{on} \quad [0, 1] \quad \text{for} \quad i = 0, \ 1, \ \cdots, n - 2. \]

Therefore,
\[ f^*(t, u(t), u^{(1)}(t), \cdots, u^{(n-2)}(t)) = f(t, u(t), u^{(1)}(t), \cdots, u^{(n-2)}(t)) \quad \text{on} \quad [0, 1]. \]
That is, $u(t)$ is a solution of (BVP) and satisfies
\[ u^{(i)}(t) \leq u^{(i)}(t) \leq u^{(i)}(t) \quad \text{on} \quad [0, 1] \quad \text{for} \quad i = 0, \ 1, \ \cdots, n - 2. \]

**Theorem 2.3** (Main result). **Suppose that**
\begin{itemize}
  \item[(H)] there exists a function $g \in \left([0, 1] \times (0, \infty)^{n-1}; [0, \infty)\right)$ which satisfies
    \[ f(t, 0, 0, \cdots, 0) \geq 0 \quad \text{on} \quad [0, 1] \quad (f \ \text{maybe has negative value for} \ u_i \neq 0), \]
  \item[(2.1)] $\max g_0 = A_1 \in [0, D_1)$ and $\min g_\infty = A_2 \in \left(\frac{D_2}{M}, \infty\right], \$
  \item[(2.2)] $\min g_0 = A_3 \in \left(\frac{D_3}{M}, \infty\right]$ and $\max g_\infty = A_4 \in [0, D_1),$
  \item[(2.3)] there exist two non-negative functions $h \in C([0, \infty)^{n-1}; [0, \infty))$, increasing
    with respect to $u_{n-1} \in [0, \infty)$, and $q \in C([0, 1]; [0, \infty))$ such that
    \[ \sup_{u_{n-1} \in (0, \infty)} \min_{(u_1, \cdots, u_{n-1}) \in [0, \infty)^{n-1}} \frac{g(t, u_{n-1})} {q(u_1, \cdots, u_{n-1})} > 1, \]
\end{itemize}
where
\[ \max g_0 := \lim_{u_{n-1} \to 0^+} \max_{t \in [0, 1]} \frac{g(t, u_1, u_2, \cdots, u_{n-1})} {u_{n-1}}, \]
\[ \min g_0 := \lim_{u_{n-1} \to 0^+} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{g(t, u_1, u_2, \cdots, u_{n-1})} {u_{n-1}}, \]
\[ \max g_\infty := \lim_{u_{n-1} \to \infty} \max_{t \in [0, 1]} \frac{g(t, u_1, u_2, \cdots, u_{n-1})} {u_{n-1}}, \]
\[ \min g_\infty := \lim_{u_{n-1} \to \infty} \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{g(t, u_1, u_2, \cdots, u_{n-1})} {u_{n-1}}, \]
\[ \left( \int_{0}^{1} k(s, s) ds \right)^{-1} := D_1 = \frac{6\rho}{6\delta\beta + 3\gamma\beta + \alpha\gamma + 3\alpha\delta}, \]
\[ \left( \int_{\frac{1}{2}}^{\frac{3}{2}} k(\frac{1}{2}, s) ds \right)^{-1} := D_2 = \frac{64\rho}{16\delta\beta + 6\beta\gamma + 3\alpha\gamma + 3\alpha\delta} \]
and
\[ Q := \max_{t \in [0, 1]} \int_{0}^{1} k(t, s) q(s) ds. \]
Then (BVP) has at least one non-negative solution.
Proof. From the results of Agarwal and Wong [1], [2], [3], we can see that 
(BVP\textsuperscript{**})
\[
\begin{cases}
(E\textsuperscript{**}) \quad w^{(n)}(t) + g(t, u(t), w^{(1)}(t), \ldots, w^{(n-2)}(t)) = 0 \quad \text{for } t \in (0, 1) \text{ and } n \geq 2, \\
(BC\textsuperscript{**}) \quad w^{(i)}(0) = 0, \quad 0 \leq i \leq n - 3, \\
\quad \alpha w^{(n-2)}(0) - \beta w^{(n-1)}(0) = 0, \\
\quad \gamma w^{(n-2)}(1) + \delta w^{(n-1)}(1) = 0
\end{cases}
\]
has at least one non-negative solution \( w(t) \). It is clear that \( w(t) \) and \( v(t) := 0 \) are the upper-solution and lower-solution of (BVP), respectively. From Theorem 2.2, we obtain the desired results. \( \square \)

Remark 2.4. For \( n = 2 \), there are many functions \( g(t, u) \) that do not satisfy
\[
\max g_0, \quad \min g_0, \quad \max g_\infty, \quad \min g_\infty \in \{0, \infty\},
\]
for example, \( g(t, u) := \frac{u^{n-1}}{1+u^n} \) \( (\max g_0 = 1 \text{ and } \min g_0 = \frac{16}{23}) \), \( g(t, u) := (t+1)\sin hu \) \( (\max g_0 = 2 \text{ and } \min g_0 = \frac{1}{2}) \), \( g(t, u) := u + t^2e^{-u} \) \( (\max g_0 = \infty, \max g_\infty = \min g_\infty = 1) \).

Therefore, our main result generalizes all the recent investigations about the existence of non-negative solutions of (BVP).

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References


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