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APPROXIMATION FROM LOCALLY FINITE-DIMENSIONAL SHIFT-INVARIANT SPACES

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Abstract. After exploring some topological properties of locally finite-dimensional shift-invariant subspaces \( S \) of \( L^p(\mathbb{R}^s) \), we show that if \( S \) provides approximation order \( k \), then it provides the corresponding simultaneous approximation order. In the case \( S \) is generated by a compactly supported function in \( L^\infty(\mathbb{R}) \), it is proved that \( S \) provides approximation order \( k \) in the \( L^p(\mathbb{R}) \)-norm with \( p > 1 \) if and only if the generator is a derivative of a compactly supported function that satisfies the Strang-Fix conditions.

1. Introduction

Let \( S \) be a linear space consisting of functions defined on \( \mathbb{R}^s \). \( S \) is said to be shift invariant if \( f(\cdot + \alpha) \) lies in \( S \) whenever \( f \) does, for every \( \alpha \in \mathbb{Z}^s \). \( S \) is said to be locally finite-dimensional if the restriction of \( S \) to any bounded subset of \( \mathbb{R}^s \) is finite-dimensional. A typical example for such spaces is the linear span \( S \) of a finite number of compactly supported functions and their shifts. That is, \( S = S_0(\Phi) := \text{span}\{\varphi(\cdot + \alpha): \alpha \in \mathbb{Z}^s, \varphi \in \Phi\} \), with \( \Phi \) a finite family of compactly supported functions. When \( \Phi \) consists of one function \( \varphi \), we denote \( S_0(\Phi) \) by \( S_0(\varphi) \). \( S_0(\Phi) \) is usually called a finitely generated shift-invariant space. It is clear that the shift-invariance and the local finite-dimension are purely algebraic properties. In this paper we shall show how to probe the (simultaneous) approximation order provided by \( S \) by means of these two algebraic properties.

Let \( m \geq 0 \) be an integer and \( k > 0 \). A subspace \( S \) of \( L^p(\mathbb{R}^s) \) is said to provide simultaneous approximation order \( (m, k) \) if

\[
\inf_{g \in S} \sum_{j=0}^{m} \sum_{|\alpha|=j} h^j \|D^\alpha(f - g(\cdot/h))\|_p \leq C_f h^k
\]

as \( h \to 0^+ \), for every \( f \in W^m_p(\mathbb{R}^s) \cap W^K_p(\mathbb{R}^s) \). Here, \( C_f \) is a constant independent of \( h \) and \( D^\alpha \) is the \( \alpha \)-order differentiation operator. By convention, \( S \) is said to provide approximation order \( k \) when it provides simultaneous approximation order \( (0, k) \). The simultaneous approximation order of shift-invariant subspaces generated by a

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finite number of functions has been of interest in Approximation Theory and Finite Element Analysis for a long time and it is well known [1], [11], [12] that $S_0(\phi)$ provides approximation order $k$ if $\phi \in L_p(\mathbb{R}^s)$ with $p \geq 1$ is compactly supported and satisfies the so-called Strang-Fix conditions of order $k$:

(i) $\hat{\phi}(0) \neq 0$;  
(ii) $D^\alpha \hat{\phi} = 0$ on $2\pi \mathbb{Z}^s \setminus 0$, for all $|\alpha| < k$.

Here, $\hat{\phi}$ denotes the Fourier transform of $\phi$. The Strang-Fix conditions have been so well reputed because they enable us to determine the approximation order provided by $S_0(\phi)$, with $\hat{\phi}(0) \neq 0$, by examining the single generator $\phi$, in spite of the fact that $S_0(\phi)$ is infinite-dimensional.

It is well known that the above-mentioned Strang-Fix conditions can also be described algebraically as follows. Denote by $\phi^* \phi$ the disk convolution mapping with $\phi$. Namely,

$$\phi^*: f \to \sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha)f(\alpha).$$

Since $\phi$ is assumed to be compactly supported, for each $x \in \mathbb{R}^s$, $\phi^* \phi(x)$ is a sum of finite number of terms. As shown in his paper [11], Schoenberg observed that $S_0(\phi)$ provides approximation order $k$ if $\phi^* \phi$ maps $\Pi_{k-1}$ onto $\Pi_{k-1}$ in the univariate case, where $\Pi_{k-1}$ is the linear space of all polynomials of degree $< k$. As is well known now, the Strang-Fix conditions of order $k$ are equivalent to that $\phi^* \phi$ maps $\Pi_{k-1}$ onto $\Pi_{k-1}$. Therefore, $S_0(\phi)$ provides approximation order $k$ if $\phi$ has the algebraic property that $\phi^* \phi$ maps $\Pi_{k-1}$ onto $\Pi_{k-1}$. In his recent paper [8], Jia has shown the following interesting fact: For any function $f$ defined on $\mathbb{R}^s$, there exists $a: \mathbb{Z}^s \to \mathbb{C}^s$ such that $f = \phi^* \phi$ if and only if $S_0(\phi)$ locally contains $f$. Here, $S_0(\phi)$ is said to locally contain $f$ if the restriction of $S_0(\phi)$ to any compact subset $B$ contains the restriction of $f$ to $B$. Therefore, if $\hat{\phi}(0) \neq 0$, then $S_0(\phi)$ provides approximation order $k$ if and only if $S_0(\phi)$ locally contains $\Pi_{k-1}$ [8]. We note that $S_0(\phi)$ locally containing $\Pi_{k-1}$ cannot guarantee it to provide approximation order $k$ in general. When $\hat{\phi}(0) \neq 0$, it is well known that there is a local approximation scheme that realizes the approximation order provided by $S_0(\phi)$ [1].

The condition that $\hat{\phi}(0) \neq 0$ has been assumed in the past study of approximation order of $S_0(\phi)$. As shown by Strang and Fix [12], for any compactly supported $\phi \in L_2(\mathbb{R}^s)$, if $S_0(\phi)$ provides an approximation order $k$ via a controlled approximation scheme, then this condition is also necessary. But $S_0(\phi)$ may provide a positive approximation order even if $\hat{\phi}(0) = 0$. One well-known example is the function

$$(1.2) \quad \phi(x) = \begin{cases} 
1, & \text{if } 0 \leq x \leq 1; \\
-1, & \text{if } 1 < x \leq 2; \\
0, & \text{else.}
\end{cases}$$

It is known that $S_0(\phi)$ indeed provides approximation order 1 in the $L_2(\mathbb{R})$-norm. In the recent paper by de Boor, DeVore, and Ron [4], the authors obtained a necessary and sufficient condition for $S_0(\phi)$ to provide approximation order $k$ in the $L_2(\mathbb{R}^s)$-norm, for any generator $\phi \in L_2(\mathbb{R}^s)$. A corresponding result of a necessary and sufficient condition under which $S_0(\phi)$ provides simultaneous approximation order $(m, k)$ has been presented in [13], for any $\phi \in W^m_2(\mathbb{R}^s)$. As shown in [13], for
any compactly supported univariate function \( \varphi \in W^m_2(\mathbb{R}) \), \( S_0(\varphi) \) provides simultaneous approximation order \((m,k)\) in the \( L_2(\mathbb{R})\)-norm if and only if there exist a neighborhood \( \Omega_\alpha \) of the origin and a constant \( C_\alpha \) such that
\[
|\hat{\varphi}(x + 2\pi \alpha)| \leq C_\alpha |x|^k |\hat{\varphi}(x)|, \quad \forall x \in \Omega_\alpha,
\]
for all \( \alpha \in \mathbb{Z}\setminus\{0\} \). In particular, this implies that if \( S_0(\varphi) \) provides approximation order \( k \), then it also provides simultaneous approximation order \((m,k)\).

In the next sections we shall prove that, for any shift-invariant subspace \( S \subset W^m_2(\mathbb{R}^n) \) that is locally finite-dimensional, if \( S \) provides approximation order \( k \), then it also provides simultaneous approximation order \((m,k)\). In the univariate case, we shall prove that, for any nontrivial compactly supported \( \varphi \in L_p(\mathbb{R}) \cap L_{p'}(\mathbb{R}) \) with \( p > 1 \), \( S_0(\varphi) \) provides approximation order \( k \) if and only if \( \varphi \) satisfies (1.3) for all \( \alpha \in \mathbb{Z}\setminus\{0\} \), where \( p' = p/(p-1) \) is the conjugate number of \( p \). In other words, in this case, \( S_0(\varphi) \) provides approximation order \( k \) if and only if
\[
D^k \hat{\varphi} = 0 \quad \text{on} \quad 2\pi \mathbb{Z}\setminus\{0\}, \forall 0 \leq \alpha < k + m,
\]
where \( m \) is the smallest integer such that \( D^m \hat{\varphi}(0) \neq 0 \). As we shall see, any compactly supported function \( \varphi \in L_p(\mathbb{R}) \) satisfies (1.4) if and only if it is the \( m \)th-order derivative of a compactly supported function that satisfies the Strang-Fix condition of order \( k + m \), with \( m \) the smallest integer such that \( D^m \hat{\varphi}(0) \neq 0 \). Therefore, among the compactly supported functions in \( L_p(\mathbb{R}) \) are only those that satisfy the Strang-Fix conditions or their derivatives the candidates for generating a shift-invariant space that may provide a positive approximation order.

2. Locally finite-dimensional shift-invariant spaces

In this section we show some nice topological properties owned by every shift-invariant subspace \( S \subset L_p(\mathbb{R}^n) \) that is locally finite-dimensional.

**Proposition 2.1.** Let \( S \subset L_p(\mathbb{R}^n) \) be a shift-invariant subspace with \( 1 \leq p \leq \infty \) such that \( S \) is locally finite-dimensional. For any linear mapping \( A : S \to L_q(\mathbb{R}^n) \), with \( 1 \leq q \leq \infty \), if \( A \) commutes with integer translations, then there exist two positive constants \( C_1 \) and \( C_2 \) such that, for all \( f \in S \),
\[
C_1 \left( \sum_{\alpha \in \mathbb{Z}^n} \min_{g \in \ker A} \| f(\cdot + \alpha) - g \|_{L_p([0,1]^n)}^q \right)^{1/q} \\
\leq \| Af \|_q \leq C_2 \left( \sum_{\alpha \in \mathbb{Z}^n} \min_{g \in \ker A} \| f(\cdot + \alpha) - g \|_{L_p([0,1]^n)}^q \right)^{1/q}
\]
when \( q < \infty \);
\[
C_1 \sup_{\alpha \in \mathbb{Z}^n} \min_{g \in \ker A} \| f(\cdot + \alpha) - g \|_{L_p([0,1]^n)} \\
\leq \| Af \| \leq C_2 \sup_{\alpha \in \mathbb{Z}^n} \min_{g \in \ker A} \| f(\cdot + \alpha) - g \|_{L_p([0,1]^n)}
\]
when \( q = \infty \), where \( \ker A := \{ f \in S : Af = 0 \text{ on } [0,1]^n \} \).

**Proof.** It is clear that \( \ker A_1 \) is a subspace of \( S \). For any \( f \in S \) and \( g \in S \), the restriction of \( Af \) to \([0,1]^n\) equals the restriction of \( Ag \) to \([0,1]^n\) if and only if
\[
A(f - g) = Af - Ag = 0 \quad \text{on } [0,1]^n.
\]
This proves that the restriction of the range of $A$ to $[0, 1]^s$ is isomorphic to the restriction of the quotient space $S/\ker A_1$ to $[0, 1]^s$. Since $S$ is locally finite-dimensional, the restrictions of $S/\ker A_1$ and $S$ to $[0, 1]^s$ are finite-dimensional. As is well known, any linear mapping on a finite-dimensional normed space is bounded and any two norms on a finite-dimensional space are equivalent. Consequently, there exist two positive constants $C_1$ and $C_2$ such that

$$(2.3) \quad C_1 \min_{g \in \ker A_1} \|f - g\|_{L_p([0, 1]^s)} \leq \|Af\|_{L_p([0, 1]^s)} \leq C_2 \min_{g \in \ker A_1} \|f - g\|_{L_p([0, 1]^s)}$$

for all $f \in S$. Since $S$ is shift invariant, $f \in S$ implies that $f(\cdot + \alpha) \in S$ for any $\alpha \in \mathbb{Z}^s$. Also we have that

$$(Af)(\cdot + \alpha) = A(f(\cdot + \alpha)), \quad \forall \alpha \in \mathbb{Z}^s$$

because $A$ commutes with integer translations. When $q = \infty$, (2.2) follows from (2.3) and the fact that $S$ is shift invariant. When $q < \infty$, it follows from (2.3) that, for any $f \in S$,

$$\int_{\mathbb{R}^s} |(Af)(x)|^q \, dx = \sum_{\alpha \in \mathbb{Z}^s} \int_{[0, 1]^s} |(Af)(x + \alpha)|^q \, dx$$

$$= \sum_{\alpha \in \mathbb{Z}^s} \int_{[0, 1]^s} |A(f(x + \alpha))|^q \, dx$$

$$\leq \sum_{\alpha \in \mathbb{Z}^s} C_2^q \min_{g \in \ker A_1} \left( \int_{[0, 1]^s} |f(x + \alpha) - g|^p \, dx \right)^{q/p}$$

$$= C_2^q \sum_{\alpha \in \mathbb{Z}^s} \min_{g \in \ker A_1} \|f(\cdot + \alpha) - g\|_{L_p([0, 1]^s)}^q.$$

Analogously we can establish the other inequality of (2.1). \hfill \Box

In the case where $p = q$ we obtain that $\|Af\|_p \leq C_2 \|f\|_p$.

**Corollary 2.2.** Let $1 \leq p \leq \infty$ and $S \subset L_p(\mathbb{R}^s)$ be a shift-invariant subspace that is locally finite-dimensional. If $A: S \rightarrow L_p(\mathbb{R}^s)$ is a linear mapping that commutes with integer translations, then $A$ is bounded.

For any polynomial of degree $n$: $p_n = \sum_{|\alpha| \leq n} a_\alpha (\cdot)^\alpha$, by $p_n(D)$ we mean the differential operator

$$p_n(D) := \sum_{|\alpha| \leq n} a_\alpha D^\alpha.$$

It is clear that $p_n(D)$ is a linear mapping from $W^m_p(\mathbb{R}^s)$ to $L_p(\mathbb{R}^s)$. For any sufficiently smooth function $f$ and any $t \in \mathbb{R}^s$,

$$D^\alpha(f(\cdot + t)) = (D^\alpha f)(\cdot + t).$$

This shows that $p_n(D)$ commutes with integer translations.

**Corollary 2.3.** Let $p_n$ be any polynomial in $s$-variable of degree $n$ and $S$ a shift-invariant subspace of $W^m_p(\mathbb{R}^s)$ that is locally finite-dimensional. Then there exists a constant $C$ such that $\|p_n(D)f\|_p \leq C\|f\|_p$ for all $f \in S$. 


It is well known that
\[
\left( \sum_{\alpha \in \mathbb{Z}^*} |a(\alpha)|^q \right)^{1/q} \leq \left( \sum_{\alpha \in \mathbb{Z}^*} |a(\alpha)|^p \right)^{1/p}
\]
if \(1 \leq p \leq q \leq \infty\) and \(a \in L_q(\mathbb{Z}^*)\). From the proof of Proposition 2.1 we obtain

**Corollary 2.4.** Let \(1 \leq p < q \leq \infty\) and \(S \subset L_p(\mathbb{R}^*)\) be a shift-invariant subspace that is locally finite-dimensional and is locally contained in \(L_q(\mathbb{R}^*)\). Then, \(S \subset L_q(\mathbb{R}^*)\) and there exists a constant \(C\) such that \(\|f\|_q \leq C\|f\|_p\) for all \(f \in S\).

**Corollary 2.5.** Let \(S \subset L_p(\mathbb{R}^*) \cap L_q(\mathbb{R}^*)\) be a shift-invariant subspace that is locally finite-dimensional, with \(1 \leq p < q \leq \infty\). Then the closure of \(S\) in \(L_p(\mathbb{R}^*)\) is contained in the closure of \(S\) in \(L_q(\mathbb{R}^*)\).

When \(S = S_0(\Phi)\) is generated by a finite number of compactly supported functions in \(L_p(\mathbb{R}^*)\), we denote by \(S_\ast(\Phi)\) the closure of \(S_0(\Phi)\) in the topology of pointwise convergence. Namely, \(f \in S_\ast(\Phi)\) if and only if there is a sequence \(s_j \in S_0(\Phi)\) such that \(\lim_{j \to \infty} s_j(x) = f(x)\) for almost every \(x \in \mathbb{R}^*\). One can verify that \(S_\ast(\Phi)\) consists of all functions of the form
\[
\sum_{\varphi \in \Phi} \varphi^\ast a_\varphi,
\]
for all \(a_\varphi: \mathbb{Z}^* \to \mathbb{C}^*\). As shown in [8], \(f\) is contained in \(S_\ast(\Phi)\) if and only if \(f\) is locally contained in \(S_0(\Phi)\). Since the topology of \(L_p(\mathbb{R}^*)\) is stronger than that of \(S_\ast(\Phi)\), we have the following corollary that was first observed by Jia.

**Corollary 2.6.** For any finite family \(\Phi\) of compactly supported functions in \(L_p(\mathbb{R}^*)\), \(S_\ast(\Phi) \cap L_p(\mathbb{R}^*)\) is a closed subspace of \(L_p(\mathbb{R}^*)\).

Since \(S_\ast(\Phi) \cap L_p(\mathbb{R}^*)\) is also shift-invariant and locally finite-dimensional, from Corollary 2.4 we obtain

**Proposition 2.7.** For any \(1 \leq p < q \leq \infty\) and any finite \(\Phi \subset L_p(\mathbb{R}^*) \cap L_q(\mathbb{R}^*)\) consisting of compactly supported functions, we have that \(S_\ast(\Phi) \cap L_p(\mathbb{R}^*) \subset S_\ast(\Phi) \cap L_q(\mathbb{R}^*)\).

For \(S_0(\Phi) \subset L_p(\mathbb{R}^*)\), denote by \(S_p(\Phi)\) the closure of \(S_0(\Phi)\) in \(L_p(\mathbb{R}^*)\). Let \(\varphi\) be a compactly supported function in \(L_2(\mathbb{R}^*)\). As proved by de Boor, DeVore, Ron [4], \(S_\ast(\varphi) \cap L_2(\mathbb{R}^*)\) is a subspace of \(S_2(\varphi)\). Since \(S_\ast(\varphi) \cap L_2(\mathbb{R}^*)\) contains \(S_0(\varphi)\) and is closed, it follows that \(S_2(\varphi) = S_\ast(\varphi) \cap L_2(\mathbb{R}^*)\). We shall show that this has an extension to the case where \(1 < p < 2\) and \(\varphi \in L_p(\mathbb{R}^*)\). Recall that \(p' = p/(p-1)\).

For the proof we need

**Proposition 2.8.** Let \(1 < p < \infty\) and \(\Phi \subset L_p(\mathbb{R}^*) \cap L_{p'}(\mathbb{R}^*)\) be a finite family of compactly supported functions. Then, \(S_{p'}(\Phi)\) can be identified with the dual space \((S_p(\Phi))^\ast\) of \(S_p(\Phi)\).

**Proof.** As \(S_p(\Phi)\) is a closed subspace of \(L_p(\mathbb{R}^*)\), we know that the dual space of \(S_p(\Phi)\) is isomorphic to the quotient space \(L_{p'}(\mathbb{R}^*)/(S_p(\Phi))^\perp\) [10]. Since \(L_p(\mathbb{R}^*)\) and \(L_{p'}(\mathbb{R}^*)\) are reflexive and \(S_p(\Phi)\) is a closed subspace of \(L_p(\mathbb{R}^*)\), we have that
\[
(S_p(\Phi))^\ast = (L_{p'}(\mathbb{R}^*)/(S_p(\Phi))^\perp)^\ast = (S_p(\Phi))^{\perp\ast} = S_p(\Phi).
\]
It suffices to prove the case where \(p \leq 2\) because, otherwise, \(p' \leq 2\) and \((S_p(\Phi))^\ast = (S_p(\Phi))^\ast = S_p(\Phi)\). First we prove that \(S_{p'}(\Phi)\) is dense in \((S_p(\Phi))^\ast\). For any
\[ f \in (S_p(\Phi))^{**} = S_p(\Phi) \text{ that annihilates } S_p'(\Phi), \quad \int_{\mathbb{R}^*} |f|^2 = 0 \text{ because } f \text{ lines in } S_p(\Phi) \subset S_p'(\Phi). \text{ So } f = 0. \text{ This proves that } S_p'(\Phi) \text{ is dense in } (S_p(\Phi))^*. \text{ Therefore, } S_p(\Phi) = (S_p(\Phi))^{**} = (S_p'(\Phi))^*. \]

(2.4) \[ S_p'(\Phi) = (S_p'(\Phi))^{**} = (S_p(\Phi))^*. \quad \square \]

**Theorem 2.9.** Let \( 1 < p \leq 2 \) and \( p' = p/(p-1) \). For any compactly supported \( \phi \in L_p'(\mathbb{R}^*), S_p(\phi) = S_*(\phi) \cap L_p(\mathbb{R}^*). \)

**Proof.** By Corollary 2.5 and Proposition 2.7,
\[ S_p(\phi) \subset S_*(\phi) \cap L_p(\mathbb{R}^*) \subset S_*(\phi) \cap L_2(\mathbb{R}^*) = S_2(\phi) \subset S_p'(\phi). \]

This implies the dual space of \( S_*(\phi) \cap L_p(\mathbb{R}^*) \) contains that of \( S_p(\phi) \). As we know, \( S_p(\phi) \) and \( S_*(\phi) \cap L_p(\mathbb{R}^*) \) both are closed in \( L_p(\mathbb{R}^*) \). From the reflexivity it follows that some closed subspace of \( S_p'(\phi) \) can be identified with \( (S_*(\phi) \cap L_p(\mathbb{R}^*))^* \). From Proposition 2.8 we know that \( S_p'(\phi) \) can be identified with the dual space of \( S_p(\phi) \). Hence, \( S_p(\phi) \) and \( S_*(\phi) \cap L_p(\mathbb{R}^*) \) have the same dual space \( S_p'(\phi) \). Thus we obtain that
\[ S_p(\phi) = (S_p(\phi))^{**} = (S_*(\phi) \cap L_p(\mathbb{R}^*))^{**} = S_*(\phi) \cap L_p(\mathbb{R}^*). \quad \square \]

3. **Application 1: Multivariate approximation**

In this section we apply the results obtained in Section 2 to obtain some results about multivariate approximation from shift-invariant spaces that are locally finite-dimensional.

**Theorem 3.1.** Let \( S \subset W_p^m(\mathbb{R}^*) \) be a shift-invariant subspace that is locally finite-dimensional. If \( S \) provides approximation order \( k \) in the \( L_p(\mathbb{R}^*) \)-norm, then \( S \) also provides simultaneous approximation order \( (m,k) \).

**Proof.** For any \( f \in W_p^m(\mathbb{R}^*) \), there exists \( s_h \in S \) such that
\[ ||f - s_h(\cdot/h)||_p \leq C_f h^k \]
with some constant \( C_f \) independent of \( h \). Let \( \psi \) be any compactly supported function in \( W_p^m(\mathbb{R}^*) \) such that \( S_0(\psi) \) provides simultaneous approximation order \( (m,k) \). For instance, we can choose \( \psi \) as a tensor product of some univariate B-spline functions [5]. Let
\[ S_+ := S + S_0(\psi). \]

That is, \( S_+ \) is the space spanned by \( S \) and \( S_0(\psi) \). It is clear that \( S_+ \) is shift invariant and locally finite-dimensional. Therefore, by Corollary 2.3, there exists a constant \( C \) independent of \( h \) such that, for any \( |\alpha| \leq m \),
\[ ||D^\alpha g(\cdot/h)||_p \leq Ch^{-|\alpha|} ||g(\cdot/h)||_p, \quad \forall g \in S_+. \]

(3.1) Since \( S_0(\psi) \) provides simultaneous approximation \( k \), there exists \( g_h \) in \( S_0(\psi) \) such that
\[ ||D^\alpha (f - g_h(\cdot/h))||_p \leq B_f h^{k-|\alpha|} \]
for all $|\alpha| \leq m$, where $B_f$ is some constant independent of $h$. As $g_h - s_h$ lies in $S_+$,

$$
\|D^\alpha(f - s_h(\cdot/h))\|_p \leq \|D^\alpha(f - g_h(\cdot/h))\|_p + \|D^\alpha(g_h(\cdot/h) - s_h(\cdot/h))\|_p \\
\leq B_fh^{k-|\alpha|} + Ch^{-|\alpha|}\|g_h(\cdot/h) - s_h(\cdot/h)\|_p \\
\leq B_fh^{k-|\alpha|} + Ch^{-|\alpha|}(\|f - g_h(\cdot/h)\|_p + \|f - s_h(\cdot/h)\|_p) \\
\leq (B_f + CB_f + CC_f)h^{k-|\alpha|}.
$$

From the above proof we obtain the following

**Corollary 3.2.** Let $S \subset W^m_p(\mathbb{R}^s)$ be a shift-invariant subspace that is locally finite-dimensional. For any $f \in W^m_p(\mathbb{R}^s) \cap W^k_p(\mathbb{R}^s)$ and any $s_h \in S_0(\Phi)$, if $\|f - s_h(\cdot/h)\|_p = O(h^k)$, then $\|D^\alpha(f - s_h(\cdot/h))\|_p = O(h^{k-|\alpha|})$ for every $|\alpha| \leq m$.

This shows that any approximation scheme having approximation order $k$ in the $L_p(\mathbb{R}^s)$-norm automatically has simultaneous approximation order $(m,k)$ in this case.

**Theorem 3.3.** Let $S \subset L_p(\mathbb{R}^s) \cap L_q(\mathbb{R}^s)$ be a shift-invariant subspace with $1 \leq p < q \leq \infty$, such that $S$ is locally finite-dimensional. If $S$ provides approximation order $k$ in the $L_p(\mathbb{R}^s)$-norm, then $S$ provides approximation order larger than or equal to $k - s(1/p - 1/q)$ in the $L_q(\mathbb{R}^s)$-norm.

**Proof.** Let $\psi$ be a compactly supported function in $L_\infty(\mathbb{R}^s)$ such that $\psi$ satisfies the Strang-Fix conditions of order $k$. Then $S_0(\psi)$ provides approximation order $k$ in any $L_r(\mathbb{R}^s)$-norm for all $1 \leq r \leq \infty$. Moreover [1], there is a local approximation scheme

$$
Q_h : W^k_p(\mathbb{R}^s) \to S(\psi) := \psi^* \ell_r(\mathbb{Z}^s)
$$

independent of $r$ such that $\|f - (Q_h f)(\cdot/h)\|_r = O(h^k)$.

Let $S_+$ be the space spanned by $S$ and $S(\psi)$. It is clear that $S_+$ is also shift invariant and locally finite-dimensional. By Corollary 2.4, there exists a constant $C$ such that $\|f\|_q \leq C\|f\|_p$, for all $f \in S_+$. For $f \in W^k_p(\mathbb{R}^s) \cap W^k_q(\mathbb{R}^s)$ and some $s_h \in S$,

$$
\|f - s_h(\cdot/h)\|_q \leq \|f - (Q_h f)(\cdot/h)\|_q + \|(Q_h f)(\cdot/h) - s_h(\cdot/h)\|_q \\
= h^{s/q}\|Q_h f - s_h\|_q + O(h^k) \\
\leq h^{s/q}C\|Q_h f - s_h\|_p + O(h^k) \\
= h^{s/q-p}C\|(Q_h f)(\cdot/h) - s_h(\cdot/h)\|_p + O(h^k) \\
\leq Ch^{-s(1/p-1/q)}(\|f - (Q_h f)(\cdot/h)\|_p + \|f - s_h(\cdot/h)\|_p) + O(h^k) \\
= O(h^{k-s(1/p-1/q)})
$$

because $S$ provides approximation order $k$ in the $L_p(\mathbb{R}^s)$-norm.

When $s(1/p - 1/q) < 1$, $k$ is an integer, and $S$ provides an integral approximation order in the $L_q(\mathbb{R}^s)$-norm, it follows that $S$ also provides approximation order $k$ in the $L_q(\mathbb{R}^s)$-norm. In particular, if $S$ is known to provide an integral approximation order in the $L_q(\mathbb{R}^s)$-norm for all $q \geq p$, $S$ providing approximation order $k$ in the $L_p(\mathbb{R}^s)$-norm with $k$ an integer implies $S$ providing (at least) approximation...
order $k$ in the $L_q(\mathbb{R}^d)$-norm, for all $q \geq p$. As we shall see, in the univariate case, the approximation order of any shift-invariant space generated by a compactly supported function must be an integer, provided $p \geq 2$.

4. Application 2: Univariate approximation

In the univariate case, there is an algebraic characterization for $S$ to provide approximation order $k$ even if $\hat{\varphi}(0) = 0$ for every $\varphi \in S \cap L_1(\mathbb{R})$, where $S$ is a shift-invariant subspace of $L_p(\mathbb{R})$ that is locally finite-dimensional. Let $\Phi \subset L_p(\mathbb{R})$ be a finite family of compactly supported functions. Recall that $S_\ell(\Phi)$ is the closure of $S_\ell(\Phi)$ in the topology of pointwise convergence. As proved by Jia [7], $S_\ell(\Phi)$ provides approximation order $k$ if and only if there is a compactly supported $\psi \in S_\ell(\Phi) \cap L_p(\mathbb{R})$ such that $\psi^* \ell$ maps $\Pi_{k-1}$ onto $\Pi_{k-1}$. Namely, $S_\ell(\Phi)$ provides approximation order $k$ if and only if there exist sequences $a_\varphi: \mathbb{Z} \to \mathbb{C}$ such that

$$
\psi = \sum_{\varphi \in \Phi} \psi^* \ell a_\varphi \in L_p(\mathbb{R})
$$

is compactly supported and satisfies the Strang-Fix conditions of order $k$. This result reveals an intrinsic property of $S_\ell(\varphi)$ that provides a positive approximation order, as well as of $\varphi$ because $S_\ell(\varphi)$ is the closure of $S_0(\varphi)$ in the topology of pointwise convergence. As pointed out in [7], this follows that the approximation order provided by $S_\ell(\Phi)$ is an integer. Of another interest is that, when $\Phi \subset L_p(\mathbb{R}) \cap L_q(\mathbb{R})$ with $p < q$, $S_\ell(\Phi)$ provides approximation order $k$ in the $L_q(\mathbb{R})$-norm if and only if it provides approximation order $k$ in the $L_p(\mathbb{R}^d)$-norm because, as we proved in Section 2, $S_\ell(\Phi) \cap L_p(\mathbb{R}^d) \subset S_\ell(\Phi) \cap L_q(\mathbb{R}^d)$ and a compactly supported function in $L_q(\mathbb{R})$ lies in $L_p(\mathbb{R})$.

As $S_\ell(\Phi)$ is an infinite-dimensional space unless it is trivial, apparently it is non-trivial to determine if $S_\ell(\Phi)$ contains a compactly supported function that satisfies the Strang-Fix conditions of order $k$, even if $\Phi$ consists of a single compactly supported function in $L_p(\mathbb{R})$. For any given compactly supported function $\varphi \in L_p(\mathbb{R})$, it is more practically interesting to have a necessary and sufficient condition on the generator $\varphi$ itself for determining the approximation order provided by $S_0(\varphi)$.

In the following we shall show that when $S = S_0(\varphi)$ with $\varphi \in L_p(\mathbb{R}) \cap L_p(\mathbb{R})$ compactly supported then $S_\ell(\varphi)$ and $S_0(\varphi)$ provide the same approximation order in the $L_p(\mathbb{R})$-norm for $p > 1$. In particular, we prove that, for any compactly supported $\varphi \in L_p(\mathbb{R}) \cap L_p(\mathbb{R})$, with $p > 1$, $S_0(\varphi)$ provides approximation order $k$ if and only if $\varphi = D^m \psi$ for some compactly supported $\psi \in W^m_p(\mathbb{R})$ that satisfies the Strang-Fix conditions of order $k + m$. In other words, $S_0(\varphi)$ provides approximation order $k$ if and only if there exists an integer $m \geq 0$ such that $D^m \hat{\varphi}(0) \neq 0$, and $D^\alpha \hat{\varphi} = 0$ on $2\pi\mathbb{Z} \setminus \{0\}$ for all $0 \leq \alpha < k + m$.

**Theorem 4.1.** Let $1 < p < \infty$ and $\varphi \in L_p(\mathbb{R}) \cap L_p(\mathbb{R})$ be compactly supported. Denote by $S_p(\varphi)$ and $S_\ell(\varphi)$ the closure of $S_0(\varphi)$ in the $L_p(\mathbb{R})$-norm and in the topology of pointwise convergence, respectively. Then $S_0(\varphi)$ provides approximation order $k$ in the $L_p(\mathbb{R})$-norm if and only if there exists a compactly supported function $\psi \in S_2(\varphi) \cap S_p(\varphi) \cap S_\ell(\varphi)$ such that $\psi$ satisfies the Strang-Fix conditions of order $k$.

**Proof.** We only need to prove the necessity because $S_0(\varphi)$ and $S_p(\varphi)$ provide the same approximation order. It is clear that if $S_0(\varphi)$ provides approximation order $k$ in the $L_p(\mathbb{R})$-norm, then so does $S_\ell(\varphi) \cap L_p(\mathbb{R})$. Therefore, $S_\ell(\varphi)$ contains a
Let \( \phi \) by induction we can prove that it suffices to prove that \( \phi \) is contained in \( S_p(\varphi) \) when \( p \geq 2 \). In the case \( p < 2 \), by Theorem 2.9 and Corollary 2.5, \( \phi \) is compactly supported function that satisfies the Strang-Fix conditions of order \( k \).

When \( p \geq 2 \), we have that \( p' \leq p \). Hence, if \( p \geq 2 \), then the condition that the compactly supported function \( \varphi \) belongs to \( L_p^2(\mathbb{R}) \) is automatically satisfied. As an immediate consequence, we obtain that the approximation order provided by \( S_0(\varphi) \) is an integer if \( \varphi \in L_p^2(\mathbb{R}) \) is compactly supported and \( p > 1 \).

**Corollary 4.2.** Let \( \varphi \in L_p^2(\mathbb{R}) \) be a compactly supported function, with \( 1 < p \leq \infty \). Then, \( S_0(\varphi) \) provides approximation order \( k \) in the \( L_p^2(\mathbb{R}) \)-norm if and only if \( S_0(\varphi) \) provides approximation order \( k \) in the \( L_2(\mathbb{R}) \)-norm.

**Proof.** It suffices to prove that \( S_0(\varphi) \) providing approximation order \( k \) in the \( L_2(\mathbb{R}) \)-norm implies \( S_0(\varphi) \) providing approximation order \( \geq k \) in the \( L_p(\mathbb{R}) \)-norm.

If \( S_2(\varphi) \) provides approximation order \( k \), then \( S_2(\varphi) = S_\ast \cap L_2(\mathbb{R}) \) contains a compactly supported function \( \varphi \) that satisfies the Strang-Fix conditions of order \( k \). When \( p > 2 \), from that \( S_2(\varphi) \subset S_p(\varphi) \) it follows that \( S_p(\varphi) \) also provides approximation order \( k \). In the case \( p < 2 \), we have that \( \varphi \in S_\ast(\varphi) \cap L_p^2(\mathbb{R}) = S_p(\varphi) \), because \( \varphi \) is compactly supported. So \( S_p(\varphi) \) provides approximation order \( k \).

As we know, for a nontrivial compactly supported \( \varphi \in L_2(\mathbb{R}) \), \( S_0(\varphi) \) provides approximation order \( k \) if and only if, for each \( \alpha \in \mathbb{Z} \setminus \{0\} \), there exist a neighborhood \( \Omega_\alpha \) of the origin and a constant \( C_\alpha \) such that (1.3) holds. So we have

**Theorem 4.3.** For any nontrivial compactly supported function \( \varphi \in L_p(\mathbb{R}) \cap L_p^2(\mathbb{R}), p \) satisfying \( 1 < p \leq \infty \), \( S_0(\varphi) \) provides approximation order \( k \) if and only if, for each \( \alpha \in \mathbb{Z} \setminus \{0\} \), there exist a neighborhood \( \Omega_\alpha \) of the origin and a constant \( C_\alpha \) such that

\[
|\hat{\varphi}(x + 2\pi\alpha)| \leq C_\alpha |x|^k |\hat{\varphi}(x)|, \quad \forall x \in \Omega_\alpha.
\]

(4.1)

For any nontrivial compactly supported function \( \varphi \in L_1(\mathbb{R}) \), there exists an integer \( m \geq 0 \) such that \( D^\alpha \hat{\varphi}(0) = 0 \) for all nonnegative integers \( \alpha < m \) but \( D^m \hat{\varphi}(0) \neq 0 \). As one can verify, (4.1) is equivalent to that \( D^\beta \hat{\varphi}(2\pi\alpha) = 0 \) for all integers \( 0 \leq \beta < k + m \), where \( m \) is the smallest integer such that \( D^m \hat{\varphi}(0) \neq 0 \). When \( m > 0 \), it is clear that

\[
\varphi_1(x) := \int_{-\infty}^{x} \varphi(t) \, dt
\]

is a compactly supported continuous function and for almost every \( x \in \mathbb{R} \) we have that \( D\varphi_1(x) = \varphi(x) \). Thus we obtain

\[
\hat{\varphi}_1(x) = \frac{\hat{\varphi}(x)}{ix}, \quad \forall x \neq 0
\]

and \( \hat{\varphi}_1(0) = -iD\hat{\varphi}(0) \). When \( m \geq 1 \), define

\[
\varphi_m = \int_{-\infty}^{x} \frac{(x-t)^{m-1}}{(m-1)!} \varphi(t) \, dt.
\]

by induction we can prove that \( \varphi_m \) is compactly supported, \( D^m \varphi_m = \varphi \), and

\[
\hat{\varphi}_m(x) = \frac{\hat{\varphi}(x)}{(ix)^m}, \quad \forall x \neq 0.
\]

(4.2)
Note that \( \lim_{x \to 0} \hat{\varphi}_m(x) = (-i)^m D^m \hat{\varphi}(0) \neq 0. \)

**Corollary 4.4.** Let \( p > 1, \varphi \in L_p(\mathbb{R}) \cap L_p'(\mathbb{R}) \) be a compactly supported function, and \( m \) be the smallest integer such that \( D^m \varphi(0) \neq 0 \). Then, \( S_0(\varphi) \) provides approximation order \( k \) in the \( L_p(\mathbb{R}) \)-norm if and only if \( \varphi = D^m \psi \) for some compactly supported function \( \psi \in W^m_p(\mathbb{R}) \) that satisfies the Strang-Fix conditions of order \( k + m \).

**Corollary 4.5.** For \( p > 1 \) and any compactly supported function \( \varphi \in L_p(\mathbb{R}) \cap L_p'(\mathbb{R}) \), \( S_0(\varphi) \) provides approximation order \( k \geq 1 \) if and only if it locally contains \( \Pi_{k-1} \).

**Proof.** We only need to prove the sufficiency because the necessity has been proved by Jia [8]. Since \( S_0(\varphi) \) locally contains a nontrivial subspace \( \Pi_{k-1} \), \( \varphi \) is not trivial. So, \( \varphi = D^m \psi \) for some compactly supported \( \psi \in W^m_p(\mathbb{R}) \) that satisfies \( \psi(0) \neq 0 \). Note that \( D^m S_0(\psi) = S_0(\varphi) \). It follows that \( S_0(\psi) \) locally contains \( \Pi_{k+m-1} \). Since \( \psi(0) \neq 0 \), we know that \( S_0(\psi) \) locally contains \( \Pi_{k+m-1} \) if and only if \( \psi \) satisfies the Strang-Fix conditions of order \( k + m \). Therefore, \( S_0(\varphi) \) provides approximation order \( k \).

**Example.** Let \( \varphi \) be the function defined by (1.2). It is clear that \( \varphi \) is bounded and is the first-order derivative of the following B-spline:

\[
\psi(x) = \begin{cases} 
    x, & \text{if } 0 \leq x \leq 1; \\
    2 - x, & \text{if } 1 < x \leq 2; \\
    0, & \text{else}.
\end{cases}
\]

One can verify that \( \psi \) satisfies the Strang-Fix conditions of order 2. Thus we know that \( S_0(\varphi) \) does provide approximation order 1 in the \( L_1(\mathbb{R}) \)-norm for \( p > 1 \).

As we know, if \( \hat{\varphi}(0) = 0 \), then \( S_0(\varphi) \) cannot provide any positive approximation order in the \( L_1(\mathbb{R}) \)-norm. When \( \varphi \in L_1(\mathbb{R}) \) is compactly supported and \( \hat{\varphi}(0) \neq 0 \), it is well known that \( S_0(\varphi) \) provides approximation order \( k > 0 \) if and only if \( \varphi \) satisfies the Strang-Fix conditions of order \( k \). So the approximation order provided by \( S_0(\varphi) \) in the \( L_1(\mathbb{R}) \)-norm is an integer.

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**References**


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