THE EULER IMPLICIT/EXPLICIT SCHEME FOR THE
2D TIME-DEPENDENT NAVIER-STOKES EQUATIONS
WITH SMOOTH OR NON-SMOOTH INITIAL DATA

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Abstract. This paper considers the stability and convergence results for the Euler implicit/explicit scheme applied to the spatially discretized two-dimensional (2D) time-dependent Navier-Stokes equations. A Galerkin finite element spatial discretization is assumed, and the temporal treatment is implicit/explicit scheme, which is implicit for the linear terms and explicit for the nonlinear term. Here the stability condition depends on the smoothness of the initial data $u_0 \in H^\alpha$, i.e., the time step condition is $\tau \leq C_0$ in the case of $\alpha = 2$, $\tau |\log h| \leq C_0$ in the case of $\alpha = 1$ and $\tau h^{-2} \leq C_0$ in the case of $\alpha = 0$ for mesh size $h$ and some positive constant $C_0$. We provide the $H^2$-stability of the scheme under the stability condition with $\alpha = 0, 1, 2$ and obtain the optimal $H^1 - L^2$ error estimate of the numerical velocity and the optimal $L^2$ error estimate of the numerical pressure under the stability condition with $\alpha = 1, 2$.

1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial \Omega$ and to satisfy a further condition stated in (A1) below. We consider the time-dependent Navier-Stokes problem

$$
\begin{cases}
\frac{du}{dt} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = f, \quad \text{div} \, u = 0, \quad (x, t) \in \Omega \times (0, T]; \\
u u(x, 0) = u_0(x), \quad x \in \Omega; \quad u(x, t)|_{\partial \Omega} = 0, \quad t \in [0, T],
\end{cases}
$$

where $u = u(x, t) = (u_1(x, t), u_2(x, t))$ represents the velocity vector, $p = p(x, t)$ the pressure, $f = f(x, t)$ the prescribed body force, $u_0(x)$ the initial velocity, $\nu > 0$ the viscosity, and $T > 0$ a finite time.

There are numerous works devoted to the development of efficient schemes for the Navier-Stokes equations fully implicit, semi-implicit and implicit/explicit scheme. A key issue is the stability conditions of schemes. Usually the fully implicit schemes are unconditionally stable. However, at each time step, one has to solve a system of nonlinear equations. An explicit scheme is much easier in computation. But it suffers the severely restricted time step size from stability requirement. A popular approach is based on an implicit scheme for the linear terms and a semi-implicit
scheme or an explicit scheme for the nonlinear term. A semi-implicit scheme for the nonlinear term results in a linear system with a variable coefficient matrix of time, and an explicit treatment for the nonlinear term gives a constant matrix. Stability and convergence conditions of schemes have been studied by many authors. The main results are summarized below, where we set $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$, and $0 < h < 1$ denotes the mesh size in the spatial direction and $0 < \tau = \frac{T}{N} < 1$ denotes the step size in the time direction, which may change. However, $T > 0$ is fixed throughout this paper.

- For the Crank-Nicolson scheme (fully implicit), Heywood and Rannacher [23] proved that it is almost unconditionally stable and convergent, i.e.

\begin{equation}
\tau \leq C_0,
\end{equation}

for some positive constant $C_0$ depending on the data $(\nu, \Omega, T, u_0, f)$ in the case of $d = 2, 3$.

- For a two-step scheme (semi-implicit), He and Li [14] gave the following convergence condition:

\begin{equation}
\tau h^{-1/2} \leq C_0.
\end{equation}

- For the Crank-Nicolson extrapolation scheme (semi-implicit), He [15] has proved that (1.2) is the stability and convergence condition of the scheme in the case of $d = 2$.

- For the Crank-Nicolson/Adams-Bashforth scheme (implicit/explicit), Marion and Temam provided in [32] the following stability condition:

\begin{equation}
\tau h^{-d} \leq C_0, \quad d = 2, 3,
\end{equation}

and recently, Tone [37] proved the convergence under the condition

\begin{equation}
\tau h^{-2-d/2} \leq C_0, \quad d = 2, 3.
\end{equation}

- A modified Crank-Nicolson/Adams-Bashforth scheme (implicit/explicit) was proposed by Johnston and Liu [25], in which the nonlinear term and pressure term are discretized explicitly. They claimed in their numerical simulations that the scheme is stable under the standard stability condition

\begin{equation}
||u||_{L^\infty} \tau h^{-1} \leq 1, \quad d = 2, 3.
\end{equation}

No theoretical analysis has been given.

- For a three-step backward extrapolating scheme (implicit/explicit), Baker et al. [4] gave the convergence condition

\begin{equation}
\tau h^{-4/7} \leq C_0,
\end{equation}

in the case of $d = 2, 3$.

- Clearly, the time-step condition

\begin{equation}
\tau h^{-r} \leq C_0,
\end{equation}

for some $r > 0$ was imposed in these previous works when an implicit/explicit scheme is applied.

Recently, He and Sun [19] have improved the result of (1.8) and proved that the stability and convergence condition of the Crank-Nicolson/Adams-Bashforth scheme is (1.2).
This paper focuses on the Euler implicit/explicit scheme with a finite element approximation in spatial direction for solving the time-dependent Navier-Stokes equations in the case of \( d = 2 \), which were studied by Marion and Temam [32], Tone [37], Kim and Moin [27] and Issacson and Keller [25]. The scheme consists of using an approximation in spatial direction for solving the time-dependent Navier-Stokes problem. For the Euler implicit/explicit scheme we obtain the same estimate and a priori estimate results of the finite element solution (the Euler implicit/explicit scheme), excepting the discretization. Under the assumptions (A1) and (A2) in §2 with \( u^0 \in D(A^{2/3}) \), \( \alpha = 0, 1, 2 \) and (A3) in §3, we prove that the scheme is stable, i.e.,

\[
\sigma^{2-\alpha}(t_m)\left(\frac{u_h^m - u_h^{m-1}}{\tau}\right)^2 + \nu^2 \|A_h u_h^m\|_0^2 + ||p_h^m||_0^2 \leq \kappa, \quad 1 \leq m \leq N,
\]

when the stability condition

\[
\tau \leq C_0, \quad \alpha = 2,
\]

\[
\tau |\log h| \leq C_0, \quad \alpha = 1,
\]

\[
\tau h^{-2} \leq C_0, \quad \alpha = 0.
\]

is satisfied. Under the stability condition (1.10) with \( \alpha = 1, 2 \), we also provide the \( H^1 - L^2 \) optimal error estimate for the numerical velocity and the \( L^2 \)-optimal error estimate for the numerical pressure:

\[
\|u(t_m) - u_h^m\|_{H^1}^2 \leq \kappa(\sigma^{-2-\alpha}(t_m)\tau^2 + \sigma^{-2-\alpha}(t_m)h^4),
\]

\[
\|u(t_m) - u_h^m\|_{H^1}^2 \leq \kappa(\sigma^{-3-\alpha}(t_m)\tau^2 + \sigma^{-2-\alpha}(t_m)h^4),
\]

\[
\|p(t_m) - p_h^m\|_{L^2}^2 \leq \kappa(\sigma^{-4-\alpha}(t_m)\tau^2 + \sigma^{-3-\alpha}(t_m)h^4),
\]

for all \( 1 \leq m \leq N \). Here \( \sigma(t) = \min\{1, t\} \), \( \kappa \) is some positive constant depending on the data \( (\nu, \Omega, T, u_0, f) \), and \( A_h \) is a discrete Stokes operator.

Moreover, similar results were proved for the Euler implicit/explicit scheme which is applied to the spatial discretization based on the spectral Galerkin method by He [11] [32].

Remark 1.1. In the case of \( \alpha = 2 \), for the first order scheme (the Euler implicit/explicit scheme) we obtain the same \( H^1 \)-error bound of the numerical velocity and a better \( L^2 \)-error bound of the numerical pressure than the second order scheme (Crank-Nicolson scheme), excepting the \( L^2 \)-error estimate for the numerical velocity. Previously, Heywood and Rannacher in [23] provided the following error estimates for the numerical velocity and pressure:

\[
\|u(t_m) - u_h^m\|_{H^1}^2 \leq \kappa(\sigma^{-1}(t_m)\tau^2 + h^2), \quad 1 \leq m \leq N,
\]

\[
\|p(t_m) - p_h^m\|_{L^2}^2 \leq \kappa(\sigma^{-3}(t_m)\tau^2 + \sigma^{-1}(t_m)h^2), \quad 1 \leq m \leq N,
\]

and the \( L^2 \)-error estimate for the numerical velocity:

\[
\|u(t_m) - u_h^m\|_{L^2}^2 \leq \kappa(\sigma^{-2}(t_m)\tau^4 + h^4), \quad t_m \in (0, T], \quad 1 \leq m \leq N.
\]

This paper is organized as follows. In §2 an abstract functional setting of the Navier-Stokes problem is given together with some basic assumptions (A1) and (A2) with \( \alpha = 0, 1, 2 \). In §3 we set out our assumption (A3) concerning the finite element spaces \( X_h \) and \( M_h \), finite element Galerkin approximation in space and some properties on the trilinear form \( b(\cdot, \cdot, \cdot) \). Section 3 contains the optimal error estimate and a priori estimate results of the finite element solution \( (u_h(t), p_h(t)) \). In §4 we describe the Euler implicit/explicit scheme and prove the stability result of the scheme. In §5 we describe the dual scheme and prove its stability result. In §6 we obtain the optimal \( H^1 - L^2 \)-error estimate of the numerical velocity and the
optimal $L^2$-error estimate of the numerical pressure under the stability condition (1.10) with $\alpha = 1, 2$.

2. Functional setting of the Navier–Stokes equations

For the mathematical setting of problem (1.1), we introduce the following Hilbert spaces:

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2, \quad M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega); \int_\Omega q dx = 0 \right\}.$$  

The space $L^2(\Omega)^d$, $d = 1, 2, 4$, is associated with the usual $L^2$-scalar product $(\cdot, \cdot)$ and $L^2$-norm $\| \cdot \|_2$. The space $X$ is associated with its usual scalar product and equivalent norm

$$((u, v)) = (\nabla u, \nabla v), \quad \|u\|_X = \|\nabla u\|_0.$$  

Next, let the closed subset $V$ of $X$ be given by

$$V = \{ v \in X; \text{div} v = 0 \}$$

and denote by $H$ the closed subset of $Y$, i.e.,

$$H = \{ v \in Y; \text{div} v = 0, v \cdot n|_{\partial \Omega} = 0 \}.$$  

We refer readers to [1, 10, 22, 36] for details on these spaces. We denote the Stokes operator by $A = -P\Delta$, where $P$ is the $L^2$-orthogonal projection of $Y$ onto $H$ and the domain of $A$ by $D(A) = H^2(\Omega)^2 \cap V$. As mentioned above, we need a further assumption on $\Omega$ provided in [23].

(A1) Assume that $\Omega$ is smooth so that the unique solution $(v, q) \in (X, M)$ of the steady Stokes problem

$$-\nu \Delta v + \nabla q = g, \quad \text{div} v = 0 \quad \text{in} \ \Omega, \quad v|_{\partial \Omega} = 0,$$

for any prescribed $g \in Y$, exists and satisfies

$$\|v\|_{H^2} + \|q\|_{H^1} \leq c\|g\|_0,$$

where $c > 0$ is a generic constant depending on $\Omega$ and $\nu$, and may take different values at its different occurrences.

We remark that the validity of assumption (A1) is known (see [10, 22, 28, 36]) if $\partial \Omega$ is of $C^2$ or if $\Omega$ is a two-dimensional convex polygon. From the assumption (A1), it is well known [1, 22, 29] that

$$\|v\|_{H^2} \leq c\|Av\|_0, \quad v \in D(A),$$

$$\|v\|_0 \leq \gamma_0\|\nabla v\|_0, \quad v \in X, \quad \|\nabla v\|_0 \leq \gamma_0\|Av\|_0, \quad v \in D(A),$$

where $\gamma_0$ is a positive constant depending only on $\Omega$. We usually make the following assumption about the prescribed data for problem (1.10).

(A2) The initial velocity $u_0(x)$ and the force $f(x, t)$ are such that $u_0 \in D(A^{\alpha/2})$, $f$, $f_t$, $f_{tt} \in L^\infty(0, T; Y)$ with

$$\|A^{\alpha/2}u_0\|_0 + \sup_{0 \leq t \leq T} \{ \|f(t)\|_0 + \|f_t(t)\|_0 + \|f_{tt}(t)\|_0 \} \leq C$$

for some positive constant $C$, and $\alpha = 0, 1, 2$, where $D(A^{1/2}) = V$ and $D(A^0) = H$.

Moreover, we define the continuous bilinear forms $a(\cdot, \cdot)$ and $d(\cdot, \cdot)$ on $X \times X$ and $X \times M$, respectively, by

$$a(u, v) = \nu((u, v)), \quad u, v \in X, \quad d(v, q) = (q, \text{div} v), \quad v \in X, \quad q \in M,$$
and a trilinear form on $X \times X \times X$ by
\[
b(u, v, w) = ((u \cdot \nabla)v + \frac{1}{2}(\text{div}u)v, w) = \frac{1}{2}((u \cdot \nabla)v, w) - \frac{1}{2}((u \cdot \nabla)w, v), \quad u, v, w \in X.
\]

With the above notation, the variational formulation of problem (1.1) reads as follows: Find $(u, p) \in (X, M)$ for all $t \in [0, T]$ such that for all $(v, q) \in (X, M)$,
\[
(2.3) \quad (u_t, v) + a(u, v) - d(v, p) + d(u, q) + b(u, u, v) = (f, v),
\]
\[
(2.4) \quad u(0) = u_0.
\]

In order to proceed the theoretical and numerical analysis for the variational formulation (2.3)-(2.4), we need to introduce the following existence, uniqueness and modified regularity results.

**Theorem 2.1.** Under the assumptions (A1) and (A2), the problem (2.3)-(2.4) admits a unique solution $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ satisfying the following regularities:
\[
\|u(t)\|_0^2 + \sigma^{1-\alpha}(2-\alpha)(t)\|\nabla u(t)\|_0^2 + \sigma^{2-\alpha}(t)(\|Au(t)\|_0^2 + \|p(t)\|_0^2 + \|u_t(t)\|_0^2)
\]
\[
+ \sigma^{3-\alpha}(t)\|\nabla u_t(t)\|_0^2 \leq \kappa,
\]
\[
\int_0^t \left(\|\nabla u\|_0^2 + \sigma^{1-\alpha}(2-\alpha)(t)\|u_t\|_0^2 + \|Au\|_0^2 + \|p\|_0^2\right)ds
\]
\[
+ \int_0^t \left\{\sigma^{2-\alpha}(s)\|\nabla u_t\|_0^2 + \sigma^{3-\alpha}(s)(\|u_{tt}\|_0^2 + \|Au_t\|_0^2 + \|p_t\|_0^2)\right\}ds \leq k,
\]
for all $0 \leq t \leq T$.

**Proof.** For the existence and uniqueness of the solution in the case of $\alpha = 0$, the reader may refer to Temam [30]. For the regularity results related to $\alpha = 2$, the reader may refer to Heywood and Rannacher [22], and for the regularity results related to $\alpha = 1$, the reader may refer to Hill and Süli [24] and He [11] and He et al. [17].

The case $\alpha = 0$ has been proved in [12], except for the estimates of $\|\nabla p(t)\|_0^2$ and $\|\nabla p_t\|_0^2$. However, these can be done by using (1.1) and some nonlinear term estimates. \hfill \Box

### 3. Finite element Galerkin approximation

Let $h > 0$ be a real positive parameter. The finite element subspace $(X_h, M_h)$ of $(X, M)$ is characterized by $J_h = J_h(\Omega)$, a partitioning of $\Omega$ into triangles $K$ or quadrilaterals $K$, assumed to be uniformly regular as $h \to 0$. For further details, the reader may refer to Ciarlet [7] and Girault and Raviart [10].

We define the subspace $V_h$ of $X_h$ given by
\[
V_h = \left\{ v_h \in X_h : d(v_h, q_h) = 0, \quad \forall q_h \in M_h \right\}.
\]

Let $P_h : Y \to V_h$ denote the $L^2$-orthogonal projection defined by
\[
(P_h v, v_h) = (v, v_h), \quad v \in Y, \quad v_h \in V_h.
\]
We assume that the couple \((X_h, M_h)\) satisfies the following approximation properties:

(A3) For each \(v \in H^2(\Omega)^2 \cap X\) and \(q \in H^1(\Omega) \cap M\), there exist approximations \(\pi_h v \in X_h\) and \(\rho_h q \in M_h\) such that

\[
\|\nabla (v - \pi_h v)\|_0 \leq c_h \|Av\|_0, \quad \|q - \rho_h q\|_0 \leq c_h \|\nabla q\|_0.
\]

(3.2) For each \(v_h \in X_h\), one has the inverse inequality

\[
\|\nabla v_h\|_0 \leq c_1 h^{-1} \|v_h\|_0, \quad v_h \in X_h;
\]

and the so-called inf-sup inequality: For each \(q_h \in M_h\), there exists \(v_h \in X_h, v_h \neq 0\), such that

\[
d(v_h, q_h) \geq c_2 \|q_h\|_0 \|\nabla v_h\|_0,
\]

where \(c_1\) and \(c_2\) are positive constants depending on \(\Omega\).

We give an example of the spaces \(X_h\) and \(M_h\) such that the assumption (A3) is satisfied. Let \(\Omega\) be a convex, polygonal domain in plane and \(J_h = J_h(\Omega)\), a partitioning of \(\Omega\) into triangles \(K\), assumed to be uniformly regular as \(h \to 0\). For any nonnegative integer \(l\), we denote by \(P_l(K)\) the space of polynomials of degrees less than or equal to \(l\) on \(K\).

**Example 1** (Girault-Raviart [10]).

\[
X_h = \{v_h \in C^0(\Omega)^2 \cap X; v_h|_K \in P_2(K)^2, \forall K \in J_h\},
\]

\[
M_h = \{q_h \in M; q_h|_K \in P_0(K), \forall K \in J_h\}.
\]

**Example 2** (Bercovier-Pironneau [5]). We consider the triangulation \(J_{h/2}\) obtained by dividing each triangle of \(J_h\) into four triangles (by joining the mid-sides). We set

\[
X_h = \{v_h \in C^0(\Omega)^2 \cap X; v_h|_K \in P_1(K)^2, \forall K \in J_{h/2}\},
\]

\[
M_h = \{q_h \in C^0(\Omega) \cap M; q_h|_K \in P_1(K), \forall K \in J_{h/2}\}.
\]

The following properties are classical (see [2] [10] [22] [24]):

\[
\|\nabla P_h v\|_0 \leq c \|\nabla v\|_0, \quad v \in X,
\]

(3.6) \(\|v - P_h v\|_0 + h \|\nabla (v - P_h v)\|_0 \leq c h^2 \|Av\|_0, \quad v \in D(A),\)

(3.7) \(\|v - P_h v\|_0 \leq c h \|\nabla (v - P_h v)\|_0, \quad v \in X.\)

The standard finite element Galerkin approximation of \((2.3)-(2.4)\) based on \((X_h,M_h)\) reads as follows: Find \((u_h, p_h) \in (X_h, M_h)\) such that for all \(0 < t \leq T\) and \((v_h, q_h) \in (X_h, M_h)\),

\[
(u_{ht}, v_h) + a(u_h, v_h) - d(v_h, p_h) + d(u_h, q_h) + b(u_h, u_h, v_h) = (f, v_h),
\]

(3.8) \(u_h(0) = u_{0h} = P_h u_0.\)

With the above statements, a discrete analogue \(A_h = -P_h \Delta_h\) of the Stokes operator \(A\) is defined through the condition that \((-\Delta_h u_h, v_h) = (u_h, v_h)\) for all \(u_h, v_h \in X_h\). The restriction of \(A_h\) to \(V_h\) is invertible, with the inverse \(A_h^{-1}\). Since \(A_h^{-1}\) is self-adjoint and positive definite, we may define "discrete" Sobolev norms on \(V_h\), of any order \(r \in R\), by setting

\[
\|v_h\|_r = \|A_h^{r/2} v_h\|_0, \quad v_h \in V_h.
\]
These norms will be assumed to have various properties similar to their continuous counterparts, an assumption that implicitly imposes conditions on the structure of the spaces \( X_h \) and \( M_h \). In particular, it holds that

\[
\|v_h\|_1 = \|\nabla v_h\|_0, \quad \|v_h\|_2 = \|A_h v_h\|_0, \quad v_h \in V_h.
\]

By the way, we derive from (2.2) that

\[
(3.10) \quad \|v_h\|_0 \leq \gamma_0 \|\nabla v_h\|_0, \quad \|\nabla v_h\|_0 \leq \gamma_0 \|A_h v_h\|_0, \quad v_h \in V_h,
\]

where \( \gamma_0 > 0 \) is a constant depending only on \( \Omega \).

This section considers preliminary estimates which are useful in the error estimates of finite element solution. Some estimates of the trilinear form \( b \) are given in the following lemma and the proof can be found in [15] [16] [24].

**Lemma 3.1.** The trilinear form \( b \) satisfies the following estimates:

\[
(3.11) \quad b(u, v_h, w_h) = ((u \cdot \nabla)v_h, v_h) = -(u \cdot \nabla)w_h, v_h),
\]

\[
(3.12) \quad b(u_h, v_h, w_h) = -b(u_h, w_h, v_h),
\]

\[
(3.13) \quad |b(u_h, v_h, w_h)| + |b(v_h, u_h, w_h)| + |b(w_h, u_h, v_h)|
\]

\[
\leq c_0 |\log h|^{1/2} |u_h|_1 |v_h|_1 |w_h|_0,
\]

\[
(3.14) \quad + \frac{c_0}{2} |u_h|_1 |v_h|_0^{1/2} |v_h|_1^{1/2} |w_h|_0^{1/2} |w_h|_1^{1/2},
\]

for all \( u \in V, u_h, v_h, w_h \in X_h \) and

\[
|b(u_h, v_h, w_h)| + |b(v_h, u_h, w_h)| + |b(w_h, u_h, v_h)|
\]

\[
\leq \frac{1}{2} c_0 |A_h v_h|_0^{1/2} |v_h|_1^{1/2} |w_h|_0^{1/2} |w_h|_1^{1/2} |w_h|_0^{1/2}
\]

\[
(3.15) \quad + \frac{1}{2} c_0 |A_h v_h|_0^{1/2} |v_h|_0^{1/2} |u_h|_1 |w_h|_0^{1/2},
\]

for all \( u_h, v_h \in V_h, w_h \in X_h \), where \( c_0 > 0 \) is a constant depending only on \( \Omega \).

Before we proceed further, we need some continuous and discrete Gagliardo-Nirenberg estimates (see Temam [36] and Hill and S"uli [24]).

**Lemma 3.2.** It holds that

\[
(3.16) \quad \|v\|_{L^4} \leq c \|v\|_0^{1/2} \|\nabla v\|_0^{1/2}, \quad \forall v \in X, \quad \|\nabla v\|_{L^4} \leq c \|\nabla v\|_0^{1/2} \|A v\|_0^{1/2}, \quad \forall v \in D(A),
\]

\[
\|v_h\|_{L^\infty} \leq c \|v_h\|_0^{1/2} \|A_h v_h\|_0^{1/2}, \quad \|v_h\|_{L^\infty} \leq c \|\log h\|^{1/2} \|\nabla v_h\|_0, \quad \forall v_h \in V_h,
\]

\[
\|\nabla v_h\|_{L^4} \leq c \|\nabla v_h\|_0^{1/2} \|A_h v_h\|_0^{1/2}, \quad \forall v_h \in V_h.
\]

In order to perform our error analysis for time discretization, we recall the following smooth properties of \((u_h, p_h)\).
Theorem 3.3. Assume that assumptions (A1)-(A3) are valid. Then the finite element solution $(u_h, p_h)$ satisfies the following estimates:

$$\begin{align}
\|u_h(t)\|^2_0 + \sigma^2(1-\alpha)(2-\alpha)(t)\|\nabla u_h(t)\|^2_0 + \sigma^{2-\alpha}(t)\|A_hu_h(t)\|^2_0
&+ \int_0^t (\|\nabla u_h\|^2_0 + \sigma^2(1-\alpha)(2-\alpha)(s)\|A_hu_h(s)\|^2_0)ds \leq \kappa, \\
\sigma^{2+\tau-\alpha}(t)\|u_{ht}(t)\|^2_r &\leq \kappa, \quad r = 0, 1, 2, \\
\int_0^t \{\sigma^2(1-\alpha)(2-\alpha)(s)\|u_{ht}\|^2_0 + \sigma^{1+\tau-\alpha}(s)\|u_{ht}\|^2_r\}ds &\leq k, \quad r = 1, 2,
\end{align}$$

(3.18)

for all $0 \leq t \leq T$.

For the proof of Theorem 3.3 in the case of $\alpha = 2$, the reader is referred to Heywood and Rannacher [23] and He and Sun [19]. Theorem 3.3 with $\alpha = 1, 0$ can be proved in a manner similar to the one used in [23] [19].

Next, we can provide some bounds of the error $(u - u_h, p - p_h)$.

Theorem 3.4. Under the assumptions (A1), (A2) with $\alpha = 1, 2$ and (A3), it holds that

$$\begin{align}
\sigma^{2-\alpha}(t)\|u(t) - u_h(t)\|^2_0
&+ \kappa^2\sigma^{2-\alpha}(t)\|\nabla(u(t) - u_h(t))\|^2_0 + \sigma^{3-\alpha}(t)h^2\|p(t) - p_h(t)\|^2_0 \\
&\leq \kappa h^4,
\end{align}$$

(3.19)

for all $t \in (0, T]$.

Proof. For the case $\alpha = 2$, Heywood and Rannacher [22] have proved (3.19). For the case $\alpha = 1$, Hill and S"{u}li [21] have proved

$$\begin{align}
\sigma(t)h^2\|u(t) - u_h(t)\|^2_0 + \sigma^{2-\alpha}(t)\|\nabla(u(t) - u_h(t))\|^2_0
&+ h^2\int_0^t \|\nabla(u - u_h)\|^2_0ds \\
&\leq \kappa h^4,
\end{align}$$

(3.20)

for all $t \in (0, T]$.

Hence, it is sufficient to prove

$$\begin{align}
\sigma(t)\|p(t) - p_h(t)\|_0 \leq \kappa h, \quad \forall t \in (0, T],
\end{align}$$

(3.21)

for $\alpha = 1$.

We set $e_h = P_hu - u_h$ and subtract (3.18) from (3.23) with $v = v_h$ to obtain

$$\begin{align}
(u_t - u_{ht}, v_h) + a(u - u_h, v_h) - d(v_h, p - p_h) + b(u, u - u_h, v_h)
&+ b(u - u_h, u, v_h) - b(u - u_h, u - u_h, v_h) = 0, \quad \forall v_h \in X_h.
\end{align}$$

(3.22)

Taking $v_h = 2e_{ht} \in V_h$ in (3.22) yields

$$\begin{align}
2\|e_{ht}\|^2_0 + 2\frac{d}{dt}\|\nabla(u - u_h)\|^2_0 + 2b(u, u - u_h, e_{ht})
&+ 2b(u - u_h, u, e_{ht}) - 2b(u - u_h, u - u_h, e_{ht})
\end{align}$$

(3.23)

$$\begin{align}
= 2\alpha(u - u_h, u_t - P_hu_t) + 2\frac{d}{dt}(e_h, p - p_h) - 2d(e_h, p_t - p_h).
\end{align}$$
Due to (2.2), (3.2), (3.6) and Lemmas 3.1 and 3.2, we have

\[ 2a(u - u_h, u_t - P_h u_t) \leq 2\nu \| \nabla (u - u_h) \|_0 \| \nabla (u_t - P_h u_t) \|_0 \]
\[ \leq ch \| \nabla (u - u_h) \|_0 \| Au \|_0 , \]

\[ 2|d(e_h, p_t - \rho_h p_t)| \leq 2\sqrt{2} \| \nabla e_h \|_0 \| p_t - \rho_h p_t \|_0 \]
\[ \leq ch \| \nabla (u - u_h) \|_0 + h \| Au \|_0 \| \nabla p_t \|_0 , \]
\[ 2|b(u, u - u_h, e_{ht})| \leq 4\|u\|_L \| \nabla (u - u_h) \|_0 + \| \nabla u \|_L \| u - u_h \|_L \| e_{ht} \|_0 \]
\[ \leq \frac{1}{2} \| e_{ht} \|_0^2 + ch^{-2} \| \nabla (u - u_h) \|_0^2 . \]

Combining this inequality with (3.23) gives

\[ \| e_{ht} \|_0^2 + 2\nu \frac{d}{dt} \| \nabla (u - u_h) \|_0^2 \leq 2\nu \| \nabla (e_h, p - \rho_h p) \|_0^2 + ch \| \nabla (u - u_h) \|_0 \| Au \|_0 + ch \| \nabla (u - u_h) \|_0 + h \| Au \|_0 \| \nabla p_t \|_0 \]
\[ + c(\| Au \|_0^2 + h^{-2} \| \nabla (u - u_h) \|_0^2) \| \nabla (u - u_h) \|_0^2 . \]

Multiplying (3.24) by \( \sigma(t) \), and integrating with respect to time and then using Theorem 2.1 and (3.20), we obtain

\[ \int_0^T \sigma(s) \| e_{ht} \|_0^2 ds \leq 2\sigma(t) \| e_h(t), p(t) - \rho_h p(t) \|_0^2 + 2 \int_0^T \| \nabla (u - u_h) \|_0^2 ds + \kappa h^2 \]
\[ \leq \sigma(t) h \| \| \nabla (u - u_h) \|_0 + h \| Au \|_0 \| \nabla p \|_0 \]
\[ + h \int_0^T \| \nabla (u - u_h) \|_0 + h \| Au \|_0 \| \nabla p \|_0 ds + \kappa h^2 \leq \kappa h^2 , \]

for all \( t \in (0, T] \).

Differentiating (3.22) with respect to time gives

\[ (u_t - u_{ht}, v_h) + a(u_t - u_{ht}, v_h) - d(v_h, p_t - \rho_h p_t) + b(u_t, u - u_h, v_h) \]
\[ + b(u - u_h, u_t, v_h) + b(u_t - u_{ht}, u_t, v_h) + b(u_t - u_{ht}, u, v_h) \]
\[ - b(u_t - u_{ht}, u - u_h, v_h) - b(u - u_h, u_t - u_{ht}, v_h) = 0 , \forall v_h \in V_h . \]

Taking \( v_h = 2e_{ht} \in V_h \) in (3.20) and using Lemma 3.1, one finds

\[ \frac{d}{dt} \| e_{ht} \|_0^2 + \nu \| \nabla (u_t - u_{ht}) \|_0^2 + \nu \| \nabla e_{ht} \|_0^2 + 2b(u_t - u_{ht}, e_{ht}) + 2b(u - u_h, u_t, e_{ht}) \]
\[ + 2b(u_t - P_h u_t, e_{ht}) + 2b(u - P_h u_t, u, e_{ht}) + 2b(e_{ht}, u, e_{ht}) \]
\[ - 2b(u_t - P_h u_t, u - u_h, e_{ht}) - 2b(u - u_h, u_t - P_h u_t, e_{ht}) \]
\[ = \nu \| \nabla (u_t - P_h u_t) \|_0^2 + 2d(e_{ht}, p_t - \rho_h p_t) . \]
Due to (3.27), (3.29), (3.30) and Lemma 3.1, we have

\[ 2|b(u_t, u - u_h, e_{ht})| + 2|b(u_t, \rho_h, u_t, e_{ht})| \]
\[ \leq 8\gamma_0\|\nabla u_t\|_0^2 \|\nabla (u - u_h)\|_0 \|\nabla e_{ht}\|_0 \]
\[ \leq \frac{\nu}{8}\|\nabla e_{ht}\|_0^2 + c\|\nabla u_t\|_0^2 \|\nabla (u - u_h)\|_0^2, \]

\[ 2|b(e_{ht}, u_h, e_{ht})| \leq 4\|\nabla e_{ht}\|_0^{3/2} \|\nabla u_h\|_0^{3/2} \|\nabla u_h\|_0^{3/2} \|\nabla u_h\|_0^2 \]
\[ \leq \frac{\nu}{8}\|\nabla e_{ht}\|_0^2 + c\|\nabla u_h\|_0^2 \|\nabla e_{ht}\|_0^2, \]

\[ 2|b(u, u_t - P_h u_t, e_{ht})| + 2|b(u_t - P_h u_t, u, e_{ht})| \]
\[ \leq 8\gamma_0\|\nabla (u_t - P_h u_t)\|_0 \|\nabla u_t\|_0 \|\nabla e_{ht}\|_0 \]
\[ \leq \frac{\nu}{8}\|\nabla e_{ht}\|_0^2 + ch^2\|\nabla u_t\|_0^2 \|Au_t\|_0^2, \]

\[ 2|b(u_t - P_h u_t, u - u_h, e_{ht})| + 2|b(u - u_h, ut - P_h u_t, e_{ht})| \]
\[ \leq 8\gamma_0\|\nabla (u_t - P_h u_t)\|_0 \|\nabla u_t - P_h u_t\|_0 \|\nabla e_{ht}\|_0 \]
\[ \leq \frac{\nu}{8}\|\nabla e_{ht}\|_0^2 + ch^2\|\nabla (u - u_h)\|_0^2 \|Au_t\|_0^2, \]

\[ \nu\|\nabla (u_t - P_h u_t)\|_0^2 + 2\|e_{ht}, p_t - \rho_h p_t\| \leq ch^2\|Au_t\|_0^2 + ch\|\nabla e_{ht}\|_0 \|\nabla p_t\|_0 \]
\[ \leq \frac{\nu}{8}\|\nabla e_{ht}\|_0^2 + ch^2\|Au_t\|_0^2 + \|\nabla p_t\|_0^2. \]

Combining (3.27) with the above estimates yields

\[ \frac{d}{dt}\|e_{ht}\|_0^2 \leq c\|u_h\|_0^2 \|\nabla u_h\|_0^2 \|e_{ht}\|_0^2 + c\|\nabla u_t\|_0^2 \|\nabla (u - u_h)\|_0^2 \]
\[ + ch^2(1 + \|\nabla u_t\|_0^2 + ||\nabla u_h\|_0^2)(||Au_t\|_0^2 + ||\nabla p_t\|_0^2). \]

(3.28)

Multiplying (3.28) by \(\sigma^2(t)\), and integrating with respect to time, we obtain

\[ \sigma^2(t)\|e_{ht}(t)\|_0^2 \leq c \int_0^t \sigma(s)(1 + \|u_h\|_0^2 \|\nabla u_h\|_0^2)\|e_{ht}\|_0^2 ds \]
\[ + c \int_0^t \sigma^2(s)\|\nabla u_t\|_0^2 \|\nabla (u - u_h)\|_0^2 ds \]
\[ + ch^2 \int_0^t \sigma^2(s)(1 + \|\nabla u_h\|_0^2 + \|\nabla u_t\|_0^2)(||Au_t\|_0^2 + ||\nabla p_t\|_0^2)ds. \]

(3.29)

Using (3.20), (3.25), Theorem 2.1 and Theorem 3.3 in (3.20), we obtain

\[ \sigma^2(t)\|e_{ht}(t)\|_0^2 \leq \kappa h^2, \]

which yields

\[ \sigma^2(t)\|u_t - u_h\|_0^2 \leq 2\sigma^2(t)\|e_{ht}(t)\|_0^2 + 2\sigma^2(t)\|u_t(t) - P_h u_t(t)\|_0^2 \]
\[ \leq 2\sigma^2(t)\|e_{ht}(t)\|_0^2 + ch^2\sigma^2(t)\|\nabla u_t\|_0^2 \leq \kappa h^2. \]

(3.30)

Finally, by using (3.22), (3.29), (5.4), (3.29) and Lemma 3.2, one finds

\[ \sigma(t)\|p_t - \rho_h p_t\|_0 \leq \sigma(t)(||\rho_h p(t) - p(t)\|_0 + ||\rho_p p(t) - p_h(t)\|_0) \]
\[ \leq c\sigma(t)\|u_t - u_h\|_0(t) + ch\sigma(t)\||\nabla p(t)\|_0 \]
\[ + c\sigma(t)(1 + \|\nabla u_t\|_0 + \|\nabla (u - u_h)\|_0)\|\nabla (u(t) - u_h(t))\|_0. \]

(3.31)

Using (3.20), (3.30) and Theorem 2.1 in (3.31), we obtain (3.21). \(\Box\)
We will frequently use a discrete version of the Gronwall lemmas used in [13] and [34].

**Lemma 3.5.** Let $C$, $\tau$, and $a_n$, $b_n$, $d_n$, for integers $n \geq 0$, be nonnegative numbers such that

\[
a_m + \tau \sum_{n=1}^{m} b_n \leq \tau \sum_{n=0}^{m-1} a_n d_n + C, \quad m \geq 1.
\]

Then

\[
a_m + \tau \sum_{n=1}^{m} b_n \leq C \exp \left( \tau \sum_{n=0}^{m-1} d_n \right), \quad m \geq 1.
\]

**Theorem 3.6.** Under the assumptions (A1), (A2) with $\alpha = 1, 2$ and (A3), $u_{htt}$ and $u_{htt}$ satisfy the following bounds:

\[
\int_{0}^{t} \sigma^{3-\alpha}(s) \|A_{ht}^{1-r/2}u_{htt}\|_0^2 ds \leq \kappa, \quad r = 0, 1, 2, \quad \alpha = 1 \text{ or } r = 0, 1, \quad \alpha = 2,
\]

\[
\sigma^{4-\alpha}(t) \|u_{htt}(t)\|_0^2 + \int_{0}^{t} \sigma^{4-\alpha}(s)(\|u_{htt}\|_1^2 + \|A_{ht}^{1/2}u_{htt}\|_0^2) ds \leq \kappa,
\]

for all $0 \leq t \leq T$.

**Proof.** Differentiating (3.38) with respect to $t$ gives

\[
(u_{htt}, v_h) + a(u_{htt}, v_h) + b(u_{htt}, u_h, v_h) + b(u_h, u_{ht}, v_h) = (f_t, v_h),
\]

for all $v_h \in V_h$.

In view of (3.10) and Lemma 3.1 we deduce from (3.36) that

\[
\|A_{ht}^{1-r/2}u_{htt}\|_0 \leq (\nu + c_0 \gamma_0 \|\nabla u_h\|_0) \|A_{ht}^{1-r/2}u_{ht}\|_0 + \gamma_0 \|f_t\|_0,
\]

which yields

\[
\int_{0}^{t} \sigma^{3-\alpha}(s) \|A_{ht}^{1-r/2}u_{htt}\|_0^2 ds \leq c \int_{0}^{t} (1 + \|\nabla u_h\|_0^2) \sigma^{3-\alpha}(s) \|A_{ht}^{1-r/2}u_{htt}\|_0^2 ds
\]

\[
+ c \int_{0}^{t} \|f_t\|_0^2 ds,
\]

for $r = 0, 1, 2, \quad \alpha = 1 \text{ or } r = 0, 1, \quad \alpha = 2$. Using Theorem 3.3 in (3.39) gives (3.31).

Furthermore, by differentiating (3.36) with respect to $t$ gives

\[
(u_{httt}, v_h) + a(u_{httt}, v_h) + 2b(u_{htt}, u_h, v_h) + b(u_h, u_{ht}, v_h)
\]

\[
+ b(u_h, u_{htt}, v_h) = (f_{tt}, v_h),
\]

for all $v_h \in V_h$.

Taking $v_h = 2u_{htt}$ in (3.38) and using (3.10) and Lemma 3.1, we deduce

\[
\frac{d}{dt} \|u_{httt}\|_0^2 \leq 2\nu \|u_{htt}\|_1^2 + 4b(u_{htt}, u_{ht}, u_{htt}) + 2b(u_h, u_{ht}, u_{htt})
\]

\[
\leq \frac{\nu}{4} \|u_{htt}\|_1^2 + 4\nu^{-1}\gamma_0^2 \|f_{tt}\|_0^2.
\]
In view of (3.10) and Lemma 3.1, we deduce from (3.36) that

\[ 4|b(u_{ht}, u_{ht}, u_{ht})| \leq 2c_0\gamma_0\|u_{ht}\|_1^2\|u_{ht}\|_1 \]

\[ \leq \frac{\nu}{4}\|u_{ht}\|_1^2 + 4\nu^{-1}c_0^2\gamma_0^2\|u_{ht}\|_1^4, \]

\[ 2|b(u_{htt}, u_h, u_{ht})| \leq c_0\gamma_0\|u_{htt}\|_0\|u_{ht}\|_1\|A_h u_h\|_0 \]

\[ \leq \frac{\nu}{4}\|u_{htt}\|_1^2 + \nu^{-1}c_0^2\gamma_0^2\|A_h u_h\|_0^2\|u_{htt}\|_0^2. \]

Combining these inequalities with (3.39) gives

\[ \frac{d}{dt}\|u_{htt}\|_0^2 + \nu\|u_{htt}\|_1^2 \leq 4\nu^{-1}\gamma_0^2\|f_{tt}\|_0^2 \]

(3.40) + \ \ 4\nu^{-1}c_0^2\gamma_0^2\|u_{htt}\|_1^4 + \nu^{-1}c_0^2\gamma_0^2\|A_h u_h\|_0^2\|u_{htt}\|_0^2.

Multiplying (3.40) by \( \sigma^{4-\alpha}(t) \) yields

\[ \frac{d}{dt}(\sigma^{4-\alpha}(t)\|u_{htt}\|_0^2) + \nu\sigma^{4-\alpha}(t)\|u_{htt}\|_1^2 \leq 4\nu^{-1}\gamma_0^2\|f_{tt}\|_0^2 \]

\[ + \ \ c\sigma^{4-\alpha}(t)\|u_{ht}\|_1^4 + (4 - \alpha + \sigma(t)\|A_h u_h\|_0^2)\sigma^{3-\alpha}(t)\|u_{htt}\|_0^2. \]

Integrating (3.41) from 0 to t and using (3.38) and Theorem 3.3, we deduce

\[ \sigma^{4-\alpha}(t)\|u_{htt}\|_0^2 + \nu \int_0^t \sigma^{4-\alpha}(s)\|u_{htt}\|_0^2 ds \leq \kappa, \ \ \forall t \in (0, T). \]

Finally, it follows from (3.38), (3.10) and Lemma 3.1 that

\[ \int_0^t \sigma^{4-\alpha}(s)\|A_h^{-1/2}u_{htt}\|_0^2 ds \leq c \int_0^t (1 + \|u_1\|_1^2)\sigma^{4-\alpha}(s)\|u_{htt}\|_0^2 ds \]

(3.43) + \ \ c \int_0^t (\sigma^{4-\alpha}(s)\|u_{htt}\|_1^4 + \|f_{tt}\|_0^2) ds.

Using Theorem 3.3 in (3.33), together with (3.42), gives (3.35) for \( \alpha = 1, \ 2. \)

4. THE EULER IMPLICIT/EXPLICIT SCHEME

In this section we consider the time discretization of the finite element Galerkin approximation (3.3)-(3.4). Usually for the fully implicit scheme, at each time step, one has to solve a system of nonlinear equations. An explicit scheme is much easier in computation. But it suffers the severely restricted time step size from stability requirement. A popular approach is based on an implicit scheme for the linear terms and an explicit scheme for the nonlinear term. An explicit scheme for the nonlinear term results in a linear system with a constant coefficient matrix such that the computation is easy and the time step restriction is \( \tau \leq C_0 \) which will be proved in this section and Section 6.

Let \( \tau_n = n\tau(n = 0, 1, \ldots, N), \ \tau = \frac{T}{N} \) the time step size, and \( N \) an integer. We define \( u_h^n = u_0h = P_h u_0 \) and \( (u_h^n, p_h^n) \in (X_h, M_h) \) by the Euler implicit/explicit scheme:

(4.1) \( (d_t u_h^n, v_h) + a(u_h^n, v_h) - d(v_h, p_h^n) + d(u_h^n, q_h) + b(u_h^{n-1}, u_h^{n-1}, v_h) = (f(t_n), v_h), \)

here \( d_t u_h^n = \frac{1}{\tau}(u_h^n - u_h^{n-1}). \)

We see from (3.33) and (3.40) that

\[ \|u_h^n\|_\alpha = \|u_0h\|_\alpha = \|P_h u_0\|_\alpha \leq c_\alpha \|A^{\alpha/2}u_0\|_0, \]

if \( u_0 \in D(A^{\alpha/2}) \) for some constants \( c_\alpha \) with \( \alpha = 0, 1, 2. \)
The following theorem provides the stability of the scheme (4.1).

**Theorem 4.1.** Suppose that the assumptions (A1)-(A3) are valid and $0 < \tau < 1$ satisfies the following stability condition:

\[
G_h \tau \leq \nu, \quad G_h = \begin{cases} 
4^2 \nu^{-3/2} c_0^2 \gamma_0 \kappa_1^{1/2} \kappa^2_2, & \alpha = 2, \\
4^2 c_0^2 \nu^{-1} \kappa_1 |\log h|, & \alpha = 1, \\
4^2 c_0^2 \kappa_0 h^{-2}, & \alpha = 0.
\end{cases}
\]

Then the following hold:

\[
\|u^m_h\|^2_0 + \nu \tau \sum_{n=1}^m \|u^n_h\|^2_1 \leq \kappa_0,
\]

\[
\tau \sum_{n=1}^m \sigma^{1/2}(1-\alpha)\tau \alpha (t_n) (\nu^2 \|A_h u^n_h\|^2_0 + \nu \|d_t u^n_h\|^2_1 \tau + \|d_t u^n_h\|^2_0)
\]

\[
+ \sigma^{1/2}(1-\alpha)\tau \alpha (t_m) \nu \|u^m_h\|^2_1 \leq \kappa_1,
\]

\[
\sigma^{2-\alpha}(t_m) \nu \|A_h u^n_h\|^2_0 \leq \kappa_2,
\]

for all $0 \leq m \leq N$, where $\kappa_\alpha \geq \gamma^\alpha \|A^{\alpha/2} u_0\|^2_0$ are some positive constants depending on the data $(\nu, \Omega, T, u_0, f)$.

**Proof.** First, taking $v_h = 2u^n_h \tau \in V_h$ and $v_h = A_h u^n_h \tau + \nu^{-1} d_t u^n_h \tau \in V_h$, respectively, and $q_h = 0$ in (4.1) and using (3.12) and the relation

\[
2(x - y)x = |x|^2 - |y|^2 + |x - y|^2, \quad \forall x, y \in \mathbb{R}^2,
\]

we obtain

\[
\|u^n_h\|^2_0 - \|u^{n-1}_h\|^2_0 + \|d_t u^n_h\|^2_1 \tau + \nu \|u^n_h\|^2_1 \tau + 2b(u^{n-1}_h, u^n_h, d_t u^n_h)\tau^2
\]

\[
= 2(f(t_n), u^n_h)\tau,
\]

\[
\|u^n_h\|^2_1 - \|u^{n-1}_h\|^2_1 + \|d_t u^n_h\|^2_1 \tau + \nu \|u^n_h\|^2_1 \tau + \|A_h u^n_h\|^2_0 \tau + b(u^{n-1}_h, u^n_h, A_h u^n_h + \nu^{-1} d_t u^n_h)\tau
\]

\[
= (f(t_n), A_h u^n_h + \nu^{-1} d_t u^n_h)\tau.
\]

In view of Lemma 3.7 and (3.10), it holds that

\[
2\|b(u^{n-1}_h, u^n_h, u^n_h)\|\tau = 2\|b(u^{n-1}_h, u^n_h, d_t u^n_h)\|\tau^2 \leq \frac{1}{2} G^{1/2}(u^{n-1}_h)\|u^n_h\|_1 \|d_t u^n_h\|_0 \tau^2
\]

\[
\leq \frac{1}{2} \|d_t u^n_h\|^2_0 \tau^2 + \frac{1}{4} G(u^{n-1}_h)\|u^n_h\|^2_0 \tau^2,
\]

\[
2\|f(t_n, u^n_h)\|\tau \leq \frac{\nu}{4} \|u^n_h\|^2_1 \tau + 4 \nu^{-1} \gamma^2_0 \|f(t_n)\|^2_0 \tau,
\]
and
\[ |b(u_h^{n-1}, u^n - u_h^{n-1}, A_h u_h^n + \nu^{-1} d_t u_h^n)| \tau \]
\[ \leq \frac{1}{2} G^{1/2}(u_h^{n-1}) \|d_t u_h^n\|_1 \|A_h u_h^n + \nu^{-1} d_t u_h^n\|_0 \tau^2 \]
\[ \leq \frac{1}{2} \|d_t u_h^n\|_1^2 \tau^2 + \frac{1}{4} G(u_h^{n-1}) (\|A_h u_h^n\|^2_0 + \nu^{-2} \|d_t u_h^n\|^2_0) \tau^2, \]
\[ |b(u_h^{n-1}, u_h^n, A_h u_h^n + \nu^{-1} d_t u_h^n)| \tau \]
\[ \leq c_0(\|u_h^{n-1}\|_0^{1/2} \|A_h u_h^n\|_0^{1/2} + \|u_h^{n-1}\|_1 + \|A_h u_h^n\|_0) \tau \]
\[ \times \|A_h u_h^n\|_0^{1/2} (\|A_h u_h^n\|_0 + \nu^{-1} \|d_t u_h^n\|_0) \tau \]
\[ \leq \frac{\nu}{8} \|A_h u_h^n\|_0^2 \tau + \frac{1}{8\nu} \|d_t u_h^n\|_0^2 \tau \]
\[ + 2(\frac{4}{\nu} c_0^3 (\|u_h^{n-1}\|_0^2 \|u_h^n\|_0^2 + \|u_h^{n-1}\|_1^2) \|u_h^{n-1}\|_1^2), \]
\[ |(f(t), A_h u_h^n + \nu^{-1} d_t u_h^n)| \tau \leq \frac{\nu}{8} \|A_h u_h^n\|_0^2 \tau + \frac{1}{8\nu} \|d_t u_h^n\|_0^2 \tau + 4\nu^{-1} \|f(t)\|_0^2 \tau, \]

where
(4.10) \[ G(u_h^n) = \begin{cases} 4^2 c_0^2 \|u_h^n\|_1 \|A_h u_h^n\|_0, & \alpha = 2, \\ 4^2 c_0^2 \log \|u_h^n\|_1^2, & \alpha = 1, \\ 4^2 c_0^2 h^{-2} \|u_h^n\|_0^2, & \alpha = 0. \end{cases} \]

Combining these inequalities with (4.8) and (4.9) yields
\[ \|u_h^n\|_0^2 - \|u_h^{n-1}\|_0^2 + \frac{1}{2} \|d_t u_h^n\|_0^2 \tau^2 + \nu \|u_h^n\|_1^2 \tau + \frac{1}{2} (\nu - G(u_h^{n-1}) \|u_h^n\|_1^2 \tau \]
\[ \leq 4\nu^{-1} c_0^2 \|f(t)\|_0^2 \tau, \]
\[ 2\nu \|u_h^n\|_1^2 - 2 \nu \|u_h^{n-1}\|_1^2 + \|d_t u_h^n\|_0^2 \tau^2 + \|d_t u_h^n\|_0^2 \tau + \nu^2 \|A_h u_h^n\|_0^2 \tau \]
\[ + \frac{\nu}{2} (\nu - G(u_h^{n-1}) (\|A_h u_h^n\|_0^2 + \nu^{-2} \|d_t u_h^n\|_0^2) \tau \]
\[ \leq d_{n-1} \nu \|u_h^{n-1}\|_1^2 \tau + 8\nu^{-1} \|f(t)\|_0^2 \tau, \]

where
\[ d_{n-1} = 4(\frac{4}{\nu}) c_0^3 (\|u_h^{n-1}\|_0^2 \|u_h^n\|_0^2 + \|u_h^{n-1}\|_1^2 \|u_h^n\|_0^2). \]

Now, we define \(d_t u_h^0 = \lim_{t \to 0} u_h(t)\) through (3.8), i.e.,
(4.13) \[ (d_t u_h^0, v_h) + a(u_h^0, v_h) + b(u_h^0, u_h^0, v_h) = (f(t_0), v_h), \]
for all \(v_h \in V_h\). Then, we deduce from (4.11) and (4.13) that
(4.14) \[ (d_t u_h^1, v_h) + a(d_t u_h^1, v_h) = \frac{1}{\tau} \int_{t_0}^{t_1} (f(t), v_h) dt, \]
and
\[ (d_t u_h^n, v_h) + a(d_t u_h^n, v_h) + b(d_t u_h^n, u_h^{n-1}, v_h) + b(u_h^{n-2}, d_t u_h^{n-1}, v_h) \]
\[ = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} (f(t), v_h) dt, \]

\[ \text{(4.15)} \]
for all $2 \leq n \leq N$. Hence, it follows from (4.14) that

$$
(4.16) \quad \|d_t u^n_h\|_0^2 + \|d_t u^n_h\|^2_2 + \nu \|d_t u^n_h\|^2_2 \leq \|d_t u^n_0\|_0^2 + \frac{\gamma^2}{\nu} \int_{t_0}^{t_1} \|f(t)\|^2_0 dt.
$$

Next, by taking $v_h = 2d_t u^n_h$ in (4.15) and using (4.12), we deduce

$$
\|d_t u^n_h\|_0^2 - \|d_t u^{n-1}_h\|_0^2 = \|d_t u^n_h\|^2_2 + 2\nu \|d_t u^n_h\|^2_2 + 2b(d_t u^{n-1}_h, u^{n-1}_h, d_t u^n_h)\tau^2
$$

$$
= 2\left(\int_{t_{n-1}}^{t_n} f(t, d_t u^n_h) dt\right).
$$

In view of Lemma 3.1 and (3.10), it holds that

$$
2|b(d_t u^n_h, u^{n-1}_h, d_t u^n_h)| \leq \frac{\nu}{4} \|d_t u^n_h\|^2_2 + \frac{1}{4} G(u^{n-2}_h) \|d_t u^n_h\|^2_2 + \frac{4\gamma^2}{\nu} \int_{t_{n-1}}^{t_n} \|f(t)\|^2_0 dt.
$$

Combining these inequalities with (4.17) yields

$$
\|d_t u^n_h\|_0^2 - \|d_t u^{n-1}_h\|_0^2 + \frac{1}{2} \|d_t u^n_h\|^2_2 + \nu \|d_t u^n_h\|^2_2 \leq \frac{4\gamma^2}{\nu} \int_{t_{n-1}}^{t_n} \|f(t)\|^2_0 dt.
$$

(4.18)

for all $2 \leq n \leq N$.

Next, we deduce from (4.1) and Lemma 3.1 that

$$
2\nu \|A_h u^n_h\|_0 \leq 2\|d_t u^n\|_0 + 2\|f(t_n)\|_0 + c_0 \|u^{n-1}_h\|_0^2 + \nu \|u^{n-1}_h\|_1^2 \leq 2\|d_t u^n\|_0 + 2\|f(t_n)\|_0 + c \|A_h u^{n-1}_h\|_0 + \nu \|u^{n-1}_h\|_0^2.
$$

(4.19)

Moreover, we deduce from (2.2), (3.3), (4.1) and Lemma 3.1 that

$$
\|p^n_h\|_0 \leq c_0 \|u^n_h\|_0 + c_0 \|d_t u^n\|_0 + c \|f(t_n)\|_0 + c \|u^{n-1}_h\|_0^2.
$$

(4.20)

Now, we will prove (4.3)-(1.6) by induction. For $\alpha = 0, 1, 2$, we deduce from (3.8) that

$$
G(u^n_0)\tau \leq G_h\tau \leq \nu.
$$

Due to (4.2), (4.4), (4.6) hold for $m = 0$. For $\alpha = 0, 1$, we can obtain (4.4)-(4.6) with $m = 1$ by using (4.11)-(4.12), (4.20)-(4.21). For $\alpha = 2$, (4.13) and Lemma 3.1
can yield
\[(4.22) \quad \| d_t u_h^n \|_0 \leq 2 \nu \| A_h v_h^n \|_0 + \| f(t) \|_0 + G^{1/2}(u_h^n) \| u_h^n \|_1.\]

Hence, we imply (4.3)-(4.6) with \( m = 1 \) by using (4.2), (4.11)-(4.12), (4.16) and (4.19)-(4.22). Assuming that (4.4)-(4.6) hold for \( m = 0, 1, \ldots, J \), we want to prove that they hold for \( m = J + 1 \).

**Proof of (4.4).** In view of the induction assumption and (4.3), it holds that
\[(4.23) \quad G(u_h^{n-1}) \tau \leq G_h \tau \leq \nu, \quad 1 \leq n \leq J + 1, \quad G(u_h^{n-2}) \tau \leq G_h \tau \leq \nu, \quad 2 \leq n \leq J,\]
for \( \alpha = 0, 1, 2 \). Summing (4.11) from \( n = 1 \) to \( J + 1 \) and using (4.23), we obtain
\[(4.24) \quad \| d_t u_h^n \|_1 \leq \sum_{n=0}^{J} \| d_t u_h^n \|_1^2 + \| f(t) \|_0^2 + 2 \nu \| u_h^n \|_1^2.\]

We set
\[a_n = \nu \| u_h^n \|_1^2, \quad C = 8 \nu^{-1} T \sup_{0 \leq t \leq T} \| f(t) \|_0^2 + \| d_t u_h^n \|_1^2,\]
\[b_n = \| d_t u_h^n \|_1 + \nu \| d_t u_h^n \|_1^2 \tau + \nu^2 \| A_h u_h^n \|_0^2.\]

Applying Lemma 3.5 to (4.24) and using (4.3), we obtain (4.3) with \( m = J + 1 \).

For \( \alpha = 0 \), multiplying (4.12) by \( \sigma(t_n) \) using (4.23) and noting \( \sigma(t_n) \leq \sigma(t_{n-1}) + \tau \), which will often be used later, we obtain
\[(4.25) \quad 2 \sigma(t_n) \| u_h^{n-1} \|_1^2 - 2 \sigma(t_{n-1}) \| u_h^{n-1} \|_1^2 + \sigma(t_n) \| d_t u_h^n \|_1^2 \| d_t u_h^n \|_1^2 + \nu \| d_t u_h^n \|_1^2 \tau + \nu^2 \| A_h u_h^n \|_0^2 \tau \]
\[\leq 2 \nu \| u_h^{n-1} \|_1^2 \tau + d_{n-1} \tau + \sigma(t_{n-1}) \| u_h^{n-1} \|_1^2 \tau + 8 \nu^{-1} \sup_{0 \leq t \leq T} \| f(t) \|_0^2 \tau,\]
for all \( 1 \leq n \leq J + 1 \). Summing (4.25) from \( n = 1 \) to \( n = J + 1 \), we deduce
\[(4.26) \quad \sigma(t_{J+1}) \| u_h^{J+1} \|_1^2 \leq 4 \tau \sum_{n=0}^{J} \sigma(t_n) \| d_t u_h^n \|_1^2 + 2 \nu \tau \sum_{n=0}^{J} \| u_h^n \|_1^2 \]
\[+ \quad 8 \nu^{-1} T \sup_{0 \leq t \leq T} \| f(t) \|_0^2 + 2 \nu \tau^2 d_0 \| u_h^1 \|_1^2.\]

Setting
\[a_n = \sigma(t_n) \| u_h^n \|_1^2, \quad C = 8 \nu^{-1} T \sup_{0 \leq t \leq T} \| f(t) \|_0^2 \tau,\]
\[b_n = \sigma(t_n) \| d_t u_h^n \|_1^2 + \nu \| d_t u_h^n \|_1^2 \tau + \nu^2 \| A_h u_h^n \|_0^2.\]

Applying Lemma 3.5 to (4.26) and using (4.3)-(4.4), we arrive at (4.3) for \( m = J + 1 \).
Proof of (4.6). If \( \|A_{h}u_{h}^{J+1}\|_{0} \leq \|A_{h}u_{h}^{J}\|_{0} \), then the induction assumption yields
\[
\sigma^{2-\alpha}(t)\nu^{2}\|A_{h}u_{h}^{J+1}(t)\|_{0}^{2} \leq \kappa_{2}, \quad 1 \leq J \leq N-1,
\]
for \( \alpha = 0, 1, 2 \). Hence, we always assume that
\[
\|A_{h}u_{h}^{J+1}\|_{0} \geq \|A_{h}u_{h}^{J}\|_{0}, \quad 1 \leq J \leq N-1.
\]

For \( \alpha = 2 \), summing (4.13) from \( n = 2 \) to \( n = J+1 \), adding (4.16) and using (4.1), (4.3), (4.5) and (4.23), we deduce
\[
\|d_{t}u_{h}^{J+1}\|_{0}^{2} + \tau \sum_{n=1}^{J+1} (\nu\|d_{t}u_{h}^{n}\|_{1}^{2} + \|d_{tt}u_{h}^{n}\|_{0}^{2}) 
\leq 2\left(\frac{4}{\nu}\right)^{4} c_{0}^{4} \kappa_{0}^{2} + \frac{8\kappa_{0}^{2}}{\nu} \int_{0}^{T} \|f(t)\|_{0}^{2} dt + 2\|d_{t}u_{h}^{0}\|_{0}^{2}.
\]

Thus, by combining (4.27) with (4.28) with (4.19) and (4.20) with \( n = J+1 \), we deduce (4.6) for \( m = J+1 \).

For \( \alpha = 1 \), by multiplying (4.18) by \( \sigma(t_{n}) \) and summing from \( n = 2 \) to \( n = J+1 \) and using (4.12) with \( n = 1 \), we find
\[
\sigma(t_{J+1})\|d_{t}u_{h}^{J+1}\|_{0}^{2} + \nu\tau \sum_{n=1}^{J+1} \sigma(t_{n})\|d_{t}u_{h}^{n}\|_{1}^{2} 
\leq \tau \sum_{n=1}^{J+1} (1 + 2\left(\frac{4}{\nu}\right)^{3} c_{0}^{3} \kappa_{0}^{3} \|u_{h}^{n-1}\|_{0}^{2} \|u_{h}^{n-1}\|_{1}^{2})\|d_{t}u_{h}^{n}\|_{0}^{2} 
+ \frac{4\kappa_{0}^{2}}{\nu} \int_{0}^{T} \|f(t)\|_{0}^{2} dt + 2(1 + d_{0}\tau)\nu\|u_{h}^{0}\|_{1}^{2} + 16\nu^{-1}\|f(t_{1})\|_{0}^{2}.
\]

Now, by using (4.27), (4.21) and (4.19) in (4.29), we obtain (4.6) for \( m = J+1 \).

Finally, for \( \alpha = 0 \), by multiplying (4.18) by \( \sigma^{2}(t_{n}) \), noting \( \sigma^{2}(t_{n}) \leq \sigma^{2}(t_{n-1}) + 3\sigma(t_{n-1})\tau \), which will often be used later, summing from \( n = 2 \) to \( n = J+1 \) and using (4.12) with \( n = 1 \), we find
\[
\sigma^{2}(t_{J+1})\|d_{t}u_{h}^{J+1}\|_{0}^{2} + \nu\tau \sum_{n=1}^{J+1} \sigma^{2}(t_{n})\|d_{t}u_{h}^{n}\|_{1}^{2} 
\leq \tau \sum_{n=1}^{J+1} \sigma(t_{n})(1 + 2\left(\frac{4}{\nu}\right)^{3} c_{0}^{3} \kappa_{0}^{3} \|u_{h}^{n-1}\|_{0}^{2} \|u_{h}^{n-1}\|_{1}^{2})\|d_{t}u_{h}^{n}\|_{0}^{2} 
+ \frac{4\kappa_{0}^{2}}{\nu} T \sup_{0 \leq t \leq T} \|f(t)\|_{0}^{2} + 2(1 + d_{0}\tau)\nu\|u_{h}^{0}\|_{1}^{2} + 16\nu^{-1}\|f(t_{1})\|_{0}^{2}.
\]

Hence, by (4.30), (4.20) and (4.19), we obtain (4.6) for \( m = J+1 \).

Theorem 4.2. Under the assumptions of Theorem 4.1, it holds that
\[
\sigma^{3-\alpha}(t_{m})\|d_{t}u_{h}^{m}\|_{1}^{2} + \nu\tau \sum_{n=2}^{m} \sigma^{3-\alpha}(t_{n})\|A_{h}d_{t}u_{h}^{n}\|_{0}^{2} \leq \kappa_{3},
\]
\[
\tau \sum_{n=2}^{m} \sigma^{3-\alpha}(t_{n})\|d_{tt}u_{h}^{n}\|_{0}^{2} \leq \kappa_{4}.
\]
for all $2 \leq m \leq N$ and $\alpha = 1, 2$, where $\kappa_3$ and $\kappa_4$ are some positive constants depending on the data $(\nu, \Omega, T, u_0, f)$.

Proof. First, taking $u_h = 2A_h d_t u_h^n \tau \in V_h$ in (4.15), we deduce

$$
\|d_t u_h^n\|_1^2 - \|d_t u_h^{n-1}\|_1^2 + \|d_t u_h^n\|_0^2 + 2r\|A_h d_t u_h^n\|_0^2 \tau + 2b(d_t u_h^{n-1}, u_h^{n-1}, A_h d_t u_h^n) \tau + 2b(u_h^{n-2}, d_t u_h^{n-1}, A_h d_t u_h^n) \tau \\
\leq 2 \int_{t_{n-1}}^{t_n} (f(t), A_h d_t u_h^n) dt.
$$

(4.33)

In view of Lemma 3.1 and (3.10), it holds that

1) $|b(d_t u_h^{n-1}, u_h^{n-1}, A_h d_t u_h^n)| \leq 2c_0 \gamma_0 \|A_h u_h^{n-1}\|_1 \|d_t u_h^{n-1}\|_1 \|A_h d_t u_h^n\|_0 \tau$

$\leq \nu \|A_h d_t u_h^n\|_0^2 \tau + 8\nu^{-1} c_0^2 \gamma_0^2 \|A_h u_h^{n-1}\|_1^2 \|d_t u_h^{n-1}\|_1^2 \tau$,

2) $|b(u_h^{n-2}, d_t u_h^n, A_h d_t u_h^n)| \leq 2c_0 \gamma_0 \|u_h^{n-2}\|_1 \|d_t u_h^n\|_1 \|A_h d_t u_h^n\|_0 \tau$

$\leq \nu \|A_h d_t u_h^n\|_0^2 \tau + 2 \frac{2}{\nu} \gamma_0^2 \|u_h^{n-2}\|_1^2 \|d_t u_h^n\|_1^2 \tau$,

3) $|b(u_h^{n-1}, d_t u_h^n - d_t u_{n-1}, A_h d_t u_h^n)| \leq \frac{1}{2} G^{1/2}(u_h^{n-2}) |d_t u_h^n|_1 \|A_h d_t u_h^n\|_0 \tau$

$\leq \frac{1}{2} |d_t u_h^n|_1^2 \tau + \frac{1}{2} G(u_h^{n-2}) \|A_h d_t u_h^n\|_0^2 \tau$,

2) $\int_{t_{n-1}}^{t_n} (f(t), A_h d_t u_h^n) dt \leq \frac{\nu}{8} \|A_h d_t u_h^n\|_0^2 \tau + \frac{8}{\nu} \int_{t_{n-1}}^{t_n} |f(t)|_0^2 dt$.

Combining these inequalities with (4.33) yields

$$
\|d_t u_h^n\|_1^2 - \|d_t u_h^{n-1}\|_1^2 + \nu \|A_h d_t u_h^n\|_0^2 \tau + \frac{1}{4} (\nu - G(u_h^{n-2}) \tau) \|A_h d_t u_h^n\|_0^2 \tau \\
\leq 8\nu^{-1} c_0^2 \gamma_0^2 \|A_h u_h^{n-1}\|_1^2 \|d_t u_h^{n-1}\|_1^2 \tau + 2 \frac{2}{\nu} \gamma_0 \|u_h^{n-2}\|_1^2 \|d_t u_h^n\|_1^2 \tau \\
+ \frac{1}{2} G(u_h^{n-2}) \|A_h d_t u_h^n\|_0^2 \tau \\
+ \frac{8}{\nu} \int_{t_{n-1}}^{t_n} |f(t)|_0^2 dt,
$$

(4.34)

for all $2 \leq n \leq N$. Multiplying (4.34) by $\sigma^{t_a}(t_n)$ and using (4.23), we deduce

$$
\sigma^{3-a}(t_n) \|d_t u_h^n\|_1^2 - \sigma^{3-a}(t_{n-1}) \|d_t u_h^{n-1}\|_1^2 + \nu \sigma^{3-a}(t_n) \|A_h d_t u_h^n\|_0^2 \tau \\
\leq c \sigma^{3-a}(t_{n-1}) \|A_h u_h^{n-1}\|_0^2 \|d_t u_h^{n-1}\|_1^2 \tau \\
+ c \sigma^{2-a}(t_n) \|u_h^{n-2}\|_1^2 \|d_t u_h^n\|_1^2 \tau \\
+ c \sigma^{2-a}(t_{n-1}) \|d_t u_h^{n-1}\|_1^2 \tau + c \int_{t_{n-1}}^{t_n} |f(t)|_0^2 dt,
$$

(4.35)

for all $2 \leq n \leq N$. Summing (4.35) from $n = 2$ to $n = m$ and using (4.6), we obtain (4.31).

Then, we deduce from (4.35), (3.10) and Lemma 3.1 that

$$
\|d_t u_h^n\|_0 \leq \nu \|A_h d_t u_h^n\|_0 + c \|d_t u_h^{n-1}\|_1 (\|A_h u_h^{n-1}\|_0 + H(n-3) \|A_h u_h^{n-2}\|_0) \\
+ \frac{1}{2} H(2-n) G^{1/2}(u_h^0) \|d_t u_h^n\|_1 + \tau^{-1/2} \int_{t_{n-1}}^{t_n} |f(t)|_0^2 dt)^{1/2}
$$

(4.36)
for all $2 \leq n \leq N$, where $H(t) = 1$, as $t \geq 0$ and $H(t) = 0$, as $t < 0$. Thus, we deduce from (4.36) that
\[
\sigma^{3-\alpha}(t_n)\|d_t u_H^n\|_0^2 \tau 
\leq \sigma^{3-\alpha}(t_n)\|A_h d_t u_H^n\|_0^2 \tau + H(2 - n)G(u_H^n)(2\tau)^{3-\alpha} \|d_t u_H^n\|_1^2 \tau + c \int_{t_{n-1}}^{t_n} \|f_t\|_0^2 dt 
+ c\sigma^{2-\alpha}(t_{n-1})\|d_t u_H^{n-1}\|_1^2 (\sigma(t_{n-1})\|A_h u_H^{n-1}\|_0^2 + H(n-3)\sigma(t_{n-2})\|A_h u_H^{n-2}\|_0^2) \tau.
\]
Summing the above inequality from $n = 2$ to $n = m$ and using Theorem 4.1, (4.21) and (4.31), we get (4.32).

5. Dual Euler scheme: Stability analysis

In order to derive the $L^2$-bound on the error $u_h(t_n) - u^n_H$ in the case of $\alpha = 1$, we employ a parabolic argument that has already been used in [23] for the Crank-Nicolson scheme of the time-dependent Navier-Stokes equation. Let $1 \leq m \leq N$ be given. We consider the linearized “backward” counterpart of the discrete Navier-Stokes (4.1): For $\xi^n \in V_h$, $1 \leq n \leq m$, find $\Phi_h^{n-1} \in V_h$ such that
\[
\begin{align*}
(v_h, d_t \Phi_h^n - a(v_h, \Phi_h^{n-1}) - b(u_h^n, v_h, \Phi_h^{n-1}) - b(v_h, u_h^n, \Phi_h^{n-1}) = (v_h, \xi^n),
\end{align*}
\]
for $v_h \in V_h$ with an initial value $\Phi_h^m = 0$.

Here, we need to introduce the following discrete dual Gronwall lemma provided in [11].

Lemma 5.1. Let $C > 0$ and let $a_n, b_n, d_n$, for integers $0 \leq n \leq m$, be nonnegative numbers such that
\[
\begin{align*}
a_k + \tau \sum_{n=k}^{m} b_n \leq \tau \sum_{n=k+1}^{m} d_n a_n + C, \quad 0 \leq k \leq m.
\end{align*}
\]
Then
\[
\begin{align*}
a_k + \tau \sum_{n=k}^{m} b_n \leq C \exp(\tau \sum_{n=k+1}^{m} d_n), \quad 0 \leq k \leq m,
\end{align*}
\]
where we assume that $\tau \sum_{n=m+1}^{m} d_n = 0$.

The following lemma provides the stability of the scheme (5.1).

Lemma 5.2. Under the assumptions of Theorem 4.1 the following a priori estimate holds:
\[
\|\Phi_h^k\|_1^2 + \nu \tau \sum_{n=k}^{m} \|A_h \Phi_h^n\|_0^2 \leq \kappa \tau \sum_{n=1}^{m} \|\xi^n\|_0^2,
\]
for all $0 \leq k \leq m$.

Proof. The proof follows the line of argument used in the proofs of Theorem 4.1. In view of Lemma 3.1 and (4.3), we can prove that (5.1) admits a unique solution sequence $\{\Phi_h^k\}_0^m$. 

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Moreover, by taking \( v_h = -2A_h \Phi_h^{n-1} \tau \) in (5.1), we obtain
\[
\| \Phi_h^{n-1} \|_1^2 - \| \Phi_h^n \|_1^2 + \| d_t \Phi_h^n \|_1^2 - 2 \| h \|_0 \| A_h \Phi_h^{n-1} \|_0^2 \tau + 2b(A_h \Phi_h^{n-1}, u_h^n, \Phi_h^{n-1}) \tau + 2b(u_h^n, A_h \Phi_h^{n-1}, \Phi_h^{n-1}) \tau \\
\leq \frac{\nu}{4} |A_h \Phi_h^{n-1}|_0^2 + \frac{1}{\nu} | \xi^n |_0^2 \tau.
\]
From Lemma 3.1 and (3.10), we have
\[
(5.5)
\]
Combining (5.5) with the above estimate gives
\[
2|b(A_h \Phi_h^{n-1}, u_h^n, \Phi_h^n)| \tau + 2|b(u_h^n, A_h \Phi_h^{n-1}, \Phi_h^n)| \tau \\
\leq \frac{\nu}{4} |A_h \Phi_h^{n-1}|_0^2 + \frac{2}{\nu} \| A_h u_h^n \|_0 \| \Phi_h^n \|_1 \| A_h \Phi_h^{n-1} \|_0 \tau + 2|b(u_h^n, A_h \Phi_h^{n-1}, \Phi_h^{n-1})| \tau \\
\leq \frac{1}{2} G_{1/2} (u_h^n) \| d_t \Phi_h^n \|_1 + |A_h \Phi_h^{n-1}|_0 \tau^2 \\
\leq \frac{1}{2} \| d_t \Phi_h^n \|_1^2 \tau^2 + \frac{1}{4} G(u_h^n) |A_h \Phi_h^{n-1}|_0^2 \tau^2.
\]
Combining (5.5) with the above estimate gives
\[
(5.6)
\]
for all \( 1 \leq n \leq m \). Using (4.3) and Theorem 4.1 with \( \alpha = 1, 2 \), we have
\[
(5.7)
\]
Summing (5.6) from \( k+1 \) to \( m \) and using (5.7) and Theorem 4.1 we obtain
\[
(5.8)
\]
for all \( 0 \leq k \leq m - 1 \). Applying Lemma 5.1 to (5.8) and using Theorem 4.1 yields (5.4).

6. Error analysis

In this section, we establish the \( H^1 \)- and \( L^2 \)-bounds of the error \( e^n = u_h(t_n) - u_h^n \) and the \( L^2 \)-bound of the error \( e^n = p_h(t_n) - p_h^n \) for all \( 1 \leq n \leq N \). To do this, we take \( t = t_n \) in (5.8) and note
\[
(6.1)
\]
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Subtracting (4.1) from (6.1), we obtain
\[ (d_i e^n, v_h) + a(e^n, v_h) - d(v_h, \eta^n) + d(e^n, q_h) + b(e^n, u_h(t_n), v_h) + b(u_h^n, v_h) = (E_n, v_h), \]
for all \((v_h, q_h) \in (X_h, M_h)\)
\[
(E_n, v_h) = -\frac{1}{\tau} \int_{t_{n-1}}^{t_n} (u_{hit}(t), v_h) dt 
+ b(u_{h}^{n-1} - u_h^n, u_{h}^{n-1}, v_h) + b(u_h^n, u_h^{n-1} - u_h^n, v_h). \]

**Lemma 6.1.** Under the assumptions of Theorem 4.1 with \(\alpha = 1, 2\), the error \(E_n\) satisfies the following bounds:
\[
\tau \sum_{n=1}^{m} \| A_h^{-1} P_h E_n \|_0^2 \leq \kappa \tau^2, \tag{6.4}
\]
\[
\tau \sum_{n=1}^{m} \| A_h^{-1/2} P_h E_n \|_0^2 \leq \kappa \tau^\alpha, \tag{6.5}
\]
\[
\tau \sum_{n=1}^{m} \sigma^{2-\alpha}(t_n) \| A_h^{-1/2} P_h E_n \|_0^2 \leq \kappa \tau^2, \tag{6.6}
\]
for all \(1 \leq m \leq N\), and
\[
\sigma^{\alpha}(t_m) \| E_m \|_0^2 + \tau \sum_{n=2}^{m} \sigma^{3-\alpha}(t_n) \| E_n \|_0^2 \leq \kappa \tau^2, \quad 2 \leq m \leq N, \tag{6.7}
\]
\[
\tau \sum_{n=3}^{m} \sigma^{4-\alpha}(t_n) \| A_h^{-1/2} P_h d_t E_n \|_0^2 \leq \kappa \tau^2, \quad 3 \leq m \leq N. \tag{6.8}
\]

**Proof.** First, it follows from (6.3), (3.10) and Lemma 3.1 that
\[
\| A_h^{-1} P_h E_n \|_0^2 \tau \leq c \tau^2 \int_{t_{n-1}}^{t_n} \| A_h^{-1} u_{hit} \|_0^2 dt 
+ c \| d_t u_h^n \|_0^2 (\| u_h^n \|_1^2 + \| u_h^{n-1} \|_1^2) \tau, \tag{6.9}
\]
\[
\| A_h^{-1/2} P_h E_n \|_0^2 \tau \leq c \tau^\alpha \int_{t_{n-1}}^{t_n} \sigma^{2-\alpha}(t) \| A_h^{-1/2} u_{hit} \|_0^2 dt 
+ c \| d_t u_h^n \|_0^2 (\| u_h^n \|_1^2 + \| u_h^{n-1} \|_1^2) \tau, \tag{6.10}
\]
\[
\sigma^{2-\alpha}(t_n) \| A_h^{-1/2} P_h E_n \|_0^2 \tau \leq c \tau^2 \int_{t_{n-1}}^{t_n} \sigma^{2-\alpha}(t) \| A_h^{-1/2} u_{hit} \|_0^2 dt 
+ c \sigma^{2-\alpha}(t_n) \| d_t u_h^n \|_0^2 (\| u_h^n \|_1^2 + \| u_h^{n-1} \|_1^2) \tau. \tag{6.11}
\]

Summing (6.9), (6.10) and (6.11) from 1 to \(m\), respectively, noting \(\tau^2 \leq \sigma^{2-\alpha}(t_n) \tau^\alpha\) and using (2.2), Theorem 3.6 and Theorem 4.1, we deduce (6.4)–(6.6) for \(\alpha = 1, 2\).

Next, by using (3.10) and Lemma 3.1 we deduce from (6.3) that
\[
\| E_n \|_0 \leq c \tau^{-1/2} \int_{t_{n-1}}^{t_n} (t - t_{n-1})^2 \| u_{hit} \|_0^2 dt \]^{1/2} + c(\| A_h u_h^n \|_0 + \| A_h u_h^{n-1} \|_0) \| d_t u_h^n \|_0 \tau, \tag{6.12}
\]
for all $2 \leq n \leq N$. Hence, we deduce from (6.12) that

$$
\sigma^{3-\alpha(t_n)}\|E_n\|_0^2 \tau \leq c r^2 \int_{t_{n-1}}^{t_n} \sigma^{3-\alpha(t)}\|u_{httt}(t)\|_0^2 dt
$$

(6.13)

$$
+ c (\sigma(t_n))A_h u_h^{n-1}\|_0^2 + \sigma(t_{n-1})\|A_h u_h^{n-1}\|_0^2)\sigma^{2-\alpha(t_n)}\|d_t u_h^n\|_0^2 \tau^3,
$$

$$
\sigma^{1-\alpha(t_n)}\|E_n\|_0^2 \leq c r^2 \int_{t_{n-1}}^{t_n} \sigma^{3-\alpha(t)}\|u_{httt}(t)\|_0^2 dt
$$

(6.14)

$$
+ c (\sigma(t_n))A_h u_h^{n-1}\|_0^2 + \sigma(t_{n-1})\|A_h u_h^{n-1}\|_0^2)\sigma^{3-\alpha(t_n)}\|d_t u_h^n\|_0^2 \tau^2,
$$

for all $2 \leq n \leq N$. Summing (6.13) from $n = 2$ to $n = m$ and using (6.14) and Theorems 3.6, 4.1 and 4.2, we deduce (6.7).

Moreover, we deduce from (6.3) that

$$
(d_t E_n, v_h) = -\tau^{-2} \int_{t_{n-1}}^{t_n} (t - t_{n-1}) \int_{t-\tau}^{t} (u_{httt}(s), v_h) ds dt - b(d_t u_h^n, u_h^{n-1}, v_h)\tau - b(u_h^{n-1}, d_t u_h^n, v_h)\tau
$$

(6.15)

$$
- b(d_t u_h^n, d_t u_h^n, v_h)\tau - b(d_t u_h^n, d_t u_h^n, v_h)\tau,
$$

for all $3 \leq n \leq N$. Using (3.10) and Lemma 3.1, we deduce from (6.15) that

$$
\|A_h^{-1/2} P_h d_t E_n\|_0 \leq c \tau^{-3/2} \left( \int_{t_{n-1}}^{t_n} (t - t_{n-1})^2 \int_{t-\tau}^{t} u_{httt}(s) ds \right)^{1/2} + c \|d_t u_h^n\|_0 \|A_h u_h^{n-1}\|_0 \tau + c \|d_t u_h^n\|_0 \|A_h d_t u_h^n\|_0 \tau
$$

$$
+ c \|d_t u_h^{n-1}\|_0 \|A_h d_t u_h^{n-1}\|_0 \tau,
$$

which yields

$$
\sigma^{4-\alpha(t_n)}\|A_h^{-1/2} P_h d_t E_n\|_0^2 \tau \leq c r^2 \int_{t_{n-2}}^{t_n} \sigma^{4-\alpha(t)}\|A_h^{-1/2} u_{httt}(t)\|_0^2 dt
$$

(6.16)

$$
+ c \tau^3 \sigma^{3-\alpha(t_n)}\|u_{httt}(t)\|_0^2 \|A_h u_h^{n-1}\|_0^2 \tau + c \tau^3 \sigma^{4-\alpha(t_n)}\|d_t u_h^n\|_0^2 \|A_h d_t u_h^n\|_0^2
$$

$$
+ c \tau^3 \sigma^{4-\alpha(t_n)}\|d_t u_h^n\|_0^2 \|A_h d_t u_h^n\|_0^2 \tau.
$$

for all $3 \leq n \leq N$. Summing (6.16) from $n = 3$ to $m$ and using Theorems 3.6, 4.1 and 4.2, we deduce (6.8).

Lemma 6.2. Under the assumptions of Theorem 4.1 with $\alpha = 1, 2$, we have

$$
\|e_h^m\|_0^2 + \tau \sum_{n=1}^{m} (\|d_t e_h^n\|_0^2 \tau + \nu\|e_h^n\|_1^2) \leq \kappa \tau^\alpha,
$$

(6.17)

for all $1 \leq m \leq N$.

Proof. Taking $v_h = 2e^n \tau \in V_h$ and $q_h = 0$ in (6.2), we obtain

$$
\|e^n\|_0^2 - \|e^{n-1}\|_0^2 + \|d_t e^n\|_0^2 \tau^2 + 2\nu\|e^n\|_1^2 \tau + 2b(e^n, u_h^n, e^n)\tau
$$

(6.18)

$$
\leq \left(\frac{\nu}{4}\|e^n\|_1^2 \tau + 4\nu^{-1}\|A_h^{-1/2} P_h E_n\|_0^2 \tau.
$$
Using Lemma 3.1 and (3.10), one finds

\[ 2|b(e^{n-1}, u_h^n, e^n)|_\tau \leq 2c_0\gamma_0^{1/2}\|e^{n-1}\|_0^{1/2}\|e^n\|_1^{1/2}\|u_h^n\|_1 \|e^n\|_1 \tau \]

\[ \leq \frac{\nu}{4}(\|e^n\|^2 + \|e^{n-1}\|^2) + 2(\frac{2}{\nu})^3c_0^2\|u_h^n\|^2 \|e^{n-1}\|^2 \tau, \]

\[ 2|b(e^n - e^{n-1}, u_h^n, e^n)|_\tau \leq \frac{1}{2}G^{1/2}(u_h^n)\|e^n\|_1 |d_t e^n|_0 \tau^2 \]

\[ \leq \frac{1}{2}|||d_t e^n||^2_0 \tau^2 + \frac{1}{4}G(u_h^n)\|e^n\|^2_1 \tau^2. \]

Hence, by combining the above inequalities with (6.18), we obtain

\[ \|e^n\|^2_0 - \|e^{n-1}\|^2_0 + \frac{1}{2} \|d_t e^n\|^2_0 \tau^2 + \nu \|e^n\|^2_1 \tau \]

\[ \leq \frac{\nu}{4}(\|e^n\|^2 - \|e^{n-1}\|^2) + \frac{1}{4}(\nu - G(u_h^n)) \|e^n\|^2_1 \tau \]

\[ \leq 2(\frac{2}{\nu})^3c_0^2\|u_h^n\|^2 \|e^{n-1}\|^2 \tau + 4\nu^{-1}\|A_h^{-1/2}P_h E_n\|^2_0 \tau, \]

for all \(1 \leq n \leq N\). Moreover, summing (6.19) from 1 to \(m\) and using (5.7), we have

\[ \|e^m\|^2_0 + \tau \sum_{n=1}^{m} \frac{1}{2} \|d_t e^n\|^2_0 \tau + \nu \|e^n\|^2_1 \]

\[ \leq \tau \sum_{n=0}^{m-1} d_n \|e^n\|^2_0 + 4\nu^{-1}\tau \sum_{n=1}^{N} \|A_h^{-1/2}P_h E_n\|^2_0, \]

where \(d_n = 2(\frac{2}{\nu})^3c_0^2\|u_h^n\|^4 \|u_h^{n-1}\|^4\). We set

\[ a_n = \|e^n\|^2_0, \quad b_n = \frac{1}{2} \|d_t e^n\|^2_0 \tau + \nu \|e^n\|^2_1, \quad C = 4\nu^{-1}\tau \sum_{n=1}^{N} \|A_h^{-1/2}P_h E_n\|^2_0, \]

and apply Lemma 3.5 to (6.20) and use Theorem 4.1 and Lemma 6.1 to deduce (6.17).

With the aid of Lemma 6.2, we obtain the following error estimate.

**Lemma 6.3.** Under the assumptions of Theorem 3.1 with \(\alpha = 1, 2\), we have

\[ \sigma^{2-\alpha}(t_m) \|e^m\|^2_0 + \tau \sum_{n=1}^{m} \sigma^{2-\alpha}(t_n) \|e^n\|^2_1 \leq \kappa \tau^2, \]

for all \(1 \leq m \leq N\).

**Proof.** For \(\alpha = 2\), Lemma 6.2 yields (6.21). For \(\alpha = 1\), we let \(\{\Phi_h^m\}_0^m\) be the solution of (6.1), corresponding to the initial value \(\Phi_h^0 = 0\) and the right-hand side of \(\{\xi^1\}_0^m = \{e^m\}_0^m\). Then, by construction, it holds that

\[ \|e^n\|^2_0 = (e^n, d_t \Phi_h^n) - a(e^n, \Phi_h^{n-1}) \tau \]

\[ - b(u_h^n, e^n, \Phi_h^{n-1}) - (E_n, \Phi_h^{n-1}) \tau + b(e^n, e^n, \Phi_h^{n-1}) \tau. \]

Taking \(v_h = \Phi_h^{n-1} \tau\) in (6.2) and adding (6.22), we obtain

\[ \|e^n\|^2_0 = (e^n, \Phi_h^n) - (e^{n-1}, \Phi_h^{n-1}) - (E_n, \Phi_h^{n-1}) \tau + b(e^n, e^n, \Phi_h^{n-1}) \tau. \]
Summing (6.23) for $1 \leq n \leq m$ and using Lemmas 5.2, 6.1 and 6.2, we have

$$\sum_{n=1}^{m} \| e^n \|_0^2 \leq \left( \tau \sum_{n=1}^{m} \| A_h^{-1} P_h E_n \|_0^2 \right)^{1/2} \left( \tau \sum_{n=1}^{m} \| A_h \Phi_n^{-1} \|_0^2 \right)^{1/2}$$

$$+ c(\tau) \sum_{n=1}^{m} \| e^n \|_0^2 \| e^n \|_1^2 \left( \tau \sum_{n=1}^{m} \| A_h \Phi_n^{-1} \|_0^2 \right)^{1/2}$$

$$\leq \kappa \tau \left( \sum_{n=1}^{m} \| e^n \|_0^2 \right)^{1/2} \leq \frac{1}{2} \sum_{n=1}^{m} \| e^n \|_0^2 + \kappa \tau^2. \quad (6.24)$$

Next, multiplying (6.19) by $\sigma(t_n)$, we deduce

$$\sigma(t_n) \| e^n \|_0^2 - \sigma(t_{n-1}) \| e^{n-1} \|_0^2 + \sigma(t_n) \nu \| e^n \|_1^2 \tau + \frac{\nu}{4} \sigma(t_n) \| e^n \|_1^2 \tau - \sigma(t_{n-1}) \| e^{n-1} \|_0^2 \tau$$

$$\leq \| e^{n-1} \|_0^2 \tau + \frac{\nu}{4} \| e^{n-1} \|_1^2 \tau^2 + 2 \left( \frac{2}{3} \right)^3 c_0^4 \| u^n \|_1^2 \| e^{n-1} \|_0^2 \tau$$

$$+ 4 \nu^{-1} \sigma(t_n) \| A_h^{-1/2} P_h E_n \|_0^2, \quad (6.25)$$

for all $1 \leq n \leq N$. Summing (6.25) from $n = 1$ to $n = m$, we have

$$\sigma(t_m) \| e^m \|_0^2 + \nu \tau \sum_{n=1}^{m} \sigma(t_n) \| e^n \|_1^2 \tau + \frac{\nu}{4} \sigma(t_m) \| e^m \|_1^2 \tau$$

$$\leq \tau \sum_{n=1}^{m} \left( \| e^n \|_0^2 + \tau \nu \| e^n \|_1^2 + 2 \nu^{-1} \| A_h \Phi_n^{-1} \|_0^2 \right) \| e^n \|_0^2 \tau$$

$$+ 4 \nu^{-1} \sigma(t_n) \| A_h^{-1/2} P_h E_n \|_0^2.$$ 

Using (6.24), Theorem 4.1 and Lemmas 6.1 and 6.2 in the above inequality gives (6.21) for $\alpha = 1$. \hfill \Box

**Lemma 6.4.** Under the assumptions of Theorem 4.1 with $\alpha = 1, 2$, we have

$$\sigma^{3-\alpha}(t_m) \| e^m \|_1^2 + \tau \sum_{n=2}^{m} \sigma^{3-\alpha}(t_n)(\| d_t e^n \|_0^2 + \nu^2 \| A_h e^n \|_0^2) \leq \kappa \tau, \quad (6.26)$$

for all $1 \leq m \leq N$.

**Proof.** Taking $v_h = 2 A_h e^n \tau \in V_h$ and $q_h = 0$ in (6.2), we obtain

$$\| e^n \|_1^2 - \| e^{n-1} \|_1^2 + \| d_t e^n \|_1^2 \tau + 2 \nu \| A_h e^n \|_0^2 \tau + 2 b(e^n, u^n(t_n), A_h e^n) \tau$$

$$+ 2 b(u^n_h, e^n_h, A_h e^n) \tau \leq \frac{\nu}{4} \| A_h e^n \|_0^2 \tau + 4 \nu^{-1} \| E_n \|_0^2 \tau. \quad (6.27)$$

In view of Lemma 3.1 and (5.10), we have

$$2 \left| b(e^n, u_h(t_n), e^n) \right| \tau + 2 \left| b(u^n_h, e^n, A_h e^n) \right| \tau$$

$$\leq 2 c_0 \gamma_0^{1/2} \| e^n \|_1 \left( \| A_h u^n_h \|_0 + \| A_h u_h(t_n) \|_0 \| A_h e^n \|_0 \right) \tau$$

$$\leq \frac{\nu}{4} \| A_h e^n \|_0^2 \tau + \frac{2}{5} \| A_h e^n \|_0^2 \tau + c(\| A_h u^n_h \|_0^2 + \| A_h u_h(t_n) \|_0^2) \| e^n \|_1^2 \tau.$$ 

Hence, by combining the above inequality with (6.27), we obtain

$$\| e^n \|_1^2 - \| e^{n-1} \|_1^2 + \nu \| A_h e^n \|_0^2 \tau$$

$$\leq c(\| A_h u^n_h \|_0^2 + \| A_h u_h(t_n) \|_0^2) \| e^n \|_1^2 \tau + 4 \nu^{-1} \| E_n \|_0^2 \tau, \quad (6.28)$$
for all $1 \leq n \leq N$. Multiplying (6.28) by $\sigma^{3-\alpha}(t_n)$, we find
\[
\sigma^{3-\alpha}(t_m)\|e^n\|_1^2 - \sigma^{3-\alpha}(t_{n-1})\|e^{n-1}\|_1^2 + \nu\sigma^{3-\alpha}(t_n)\|A_he^n\|_0^2 \tau \\
\leq \sigma^{3-\alpha}(t_n)(\|A_hu^n_h\|_0^2 + \|A_hu_n(t_n)\|_0^2)\|e^n\|_1^2 + \nu\sigma^{3-\alpha}(t_n)\|A_he^n\|_0^2 \tau
\]
(6.29)
for all $1 \leq n \leq N$. Summing (6.29) from 2 to $m$, and using Theorems 3.3 and 4.1 and Lemmas 6.1, 6.2 and 6.3, we deduce
\[
\sigma^{3-\alpha}(t_m)\|e^m\|_1^2 + \nu\tau \sum_{n=2}^{m} \sigma^{3-\alpha}(t_n)\|A_he^n\|_0^2 \leq \kappa \tau^2,
\]
for all $1 \leq m \leq N$.

Finally, we deduce from (6.2), (3.10) and Lemma 3.1 that
\[
\sigma^{3-\alpha}(t_n)\|d_te^n\|_0^2 \leq \sigma^{3-\alpha}(t_n)(1 + \|u^n_h\|_1^2 + \|e^n\|_1^2)\|A_he^n\|_0^2 \tau \\
+ \sigma^{3-\alpha}(t_n)\|E_n\|_0^2 \tau,
\]
(6.31)
for all $2 \leq n \leq N$. Summing (6.31) from $n = 2$ to $n = m$ and using (6.30), Theorem 4.1 and Lemmas 6.1 and 6.2, we deduce (6.29).

It remains to prove the error estimate for the discrete pressure $p^n_h$. To do this, we need to estimate $d_te^n$. It follows from (6.2) that
\[
(d_te^n, v_h) + a(d_te^n, v_h) + b(d_te^n, u_n(t_n), v_h) \leq 0
\]
(6.32)
for all $v_h \in V_h$ and $1 \leq n \leq N$. Taking $v_h = 2d_te^n\tau$ in (6.32) and using (3.12), we get
\[
\|d_te^n\|_0^2 - \|d_te^{n-1}\|_0^2 + 2\nu\|d_te^n\|_1^2 \tau \leq 2(b(d_te^n, u_n(t_n), d_te^n) + 2\nu\|d_te^n\|_1^2 \tau
\]
(6.33)
\[
+ 2\nu\|d_te^n\|_1^2 \tau \leq \frac{\nu}{4}\|d_te^n\|_1^2 \tau + 4\nu^{-1}\|A_h^{-1/2}E_dE_ne^n\|_0^2 \tau.
\]
In view of (3.10) and Lemma 3.1, we deduce
\[
2(b(d_te^n, u_n(t_n), d_te^n)) \leq 2\nu\|d_te^n\|_1^2 \tau + 4\nu^{-1}\|A_h^{-1/2}E_dE_ne^n\|_0^2 \tau,
\]
\[
2(b(e^{n-1}, d_te^n)) \leq 2\nu\|d_te^{n-1}\|_1^2 \tau + 4\nu^{-1}\|A_h^{-1/2}E_dE_ne^n\|_0^2 \tau.
\]
Combining these inequalities with (6.33) gives
\[
\sigma^{4-\alpha}(t_m)\|d_te^m\|_0^2 - \sigma^{4-\alpha}(t_{n-1})\|d_te^{n-1}\|_0^2 \leq \frac{\nu}{4}\|d_te^n\|_1^2 \tau + \frac{\nu}{4}\|d_te^n\|_1^2 \tau
\]
(6.34)
\[
+ \sigma^{3-\alpha}(t_{n-1})\|A_h^{-1/2}E_dE_ne^n\|_0^2 \tau + \sigma^{4-\alpha}(t_n)\|A_h^{-1/2}E_dE_ne^n\|_0^2 \tau.
\]
Summing (6.34) from 3 to $m$ and using Theorem 3.3, Theorem 4.1, Lemma 6.1, Lemma 6.3 with $m = 1, 2$ and Lemma 6.4, we obtain
\[
\sigma^{4-\alpha}(t_m)\|d_te^m\|_0^2 \leq \kappa \tau^2, 1 \leq m \leq N.
\]
Moreover, we deduce from (6.2), (3.10) and Lemma 3.1 that
\[
\|E_1\|_0 \leq \|u_{ht}(t_1)\|_0 + \tau^{-1/2} \left( \int_{t_0}^{t_1} \|u_{ht}\|_0^2 dt \right)^{1/2} + \frac{1}{2} \left( G^{1/2}(u_h^0) + G^{1/2}(u_h^1) \right) \|d_t u_h^1\|_1 \tau,
\]
which yields
\[
\sigma^{4-\alpha}(t_1) \|E_1\|_0^2 \leq c\sigma^{4-\alpha}(t_1) \|u_{ht}(t_1)\|_0^2 + c\sigma^{3-\alpha}(t_1) \int_{t_0}^{t_1} \|u_{ht}\|_0^2 dt + c\tau^2 (G(u_h^0) + G(u_h^1)) \|d_t u_h^1\|_1^2.
\]
(6.36)

Using (5.7), Theorem 3.3 and Theorem 4.2 in (6.36), we obtain
\[
\sigma^{4-\alpha}(t_1) \|E_1\|_0^2 \leq \kappa \tau^2.
\]
(6.37)

By (3.4), (3.10), (6.2) and Lemma 3.1 we deduce
\[
\|e_m\|_0 \leq c(\|d_t e^m\|_0 + \|e^m\|_1) + c\|e^m\|_1(\|u_h(t_m)\|_1 + \|u_h^m\|_1)
\]
which together with Theorems 3.3 and 4.1 yield
\[
\sigma^{4-\alpha}(t_m) \|e_m\|_0^2 \leq \kappa \sigma^{4-\alpha}(t_m) \|d_t e^m\|_0^2 + \kappa \sigma^{3-\alpha}(t_m) \|e^m\|_1^2 + \sigma^{4-\alpha}(t_m) \|E_m\|_0^2.
\]
(6.38)

Using (6.35), (6.37), Lemma 6.1 and Lemma 6.4 in (6.38) yields
\[
\sigma^{4-\alpha}(t_m) \|e_m\|_0^2 \leq \kappa \tau^2, \quad 1 \leq m \leq N.
\]
(6.39)

Combining (6.39) with Lemma 6.3 and Lemma 6.4 yields the following error estimates results.

**Theorem 6.5.** Under the assumptions of Theorem 4.1, the following error estimates hold:
\[
\sigma^{2-\alpha}(t_m) \|u_h(t_m) - u_h^m\|_0^2 + \sigma^{3-\alpha}(t_m) \|u_h(t_m) - u_h^m\|_1^2 \leq \kappa \tau^2, \quad t_m \in (0, T],
\]
(6.40)
\[
\sigma^{4-\alpha}(t_m) \|p_h(t_m) - p_h^m\|_0^2 \leq \kappa \tau^2, \quad t_m \in (0, T].
\]
(6.41)

**Remark.** Combining Theorem 6.5 with (3.19) yields (1.11) - (1.13).

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**References**

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